# ARML Competition 2011

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# Dedication

This year's ARML Contest is dedicated to our friend and fellow author Leo Schneider (1937-2010), who passed away hours after receiving the Alfred Kalfus Founder's Award for his many contributions to ARML. At our meetings, Leo's cheery disposition made for lively and amicable discussions about all aspects of mathematics, and thousands of students benefited from his problems, solutions, and suggestions. Leo's influence extends throughout this year's contest. This influence is most noticeable in the Power Question, which was inspired by a problem Leo submitted last spring. Much more important, however, is what our generation of authors learned from him, both as an author of competitions in which we participated, and as a cherished colleague. We miss Leo, his mathematics, and his friendship.

# 1 Individual Problems

**Problem 1.** Compute the $2011^{\text{th}}$ smallest positive integer $N$ that gains an extra digit when doubled.

**Problem 2.** In triangle $ABC$, $C$ is a right angle and $M$ is on $\overline{AC}$. A circle with radius $r$ is centered at $M$, is tangent to $\overline{AB}$, and is tangent to $\overline{BC}$ at $C$. If $AC = 5$ and $BC = 12$, compute $r$.

**Problem 3.** The product of the first five terms of a geometric progression is 32. If the fourth term is 17, compute the second term.

**Problem 4.** Polygon $A_1A_2 \ldots A_n$ is a regular $n$-gon. For some integer $k < n$, quadrilateral $A_1A_2A_kA_{k+1}$ is a rectangle of area 6. If the area of $A_1A_2 \ldots A_n$ is 60, compute $n$.

**Problem 5.** A bag contains 20 lavender marbles, 12 emerald marbles, and some number of orange marbles. If the probability of drawing an orange marble in one try is $\frac{1}{y}$, compute the sum of all possible integer values of $y$.

**Problem 6.** Compute the number of ordered quadruples of integers $(a, b, c, d)$ satisfying the following system of equations:

$$\begin{cases} abc &= 12{,}000 \\ bcd &= 24{,}000 \\ cda &= 36{,}000. \end{cases}$$

**Problem 7.** Let $n$ be a positive integer such that $\dfrac{3 + 4 + \cdots + 3n}{5 + 6 + \cdots + 5n} = \dfrac{4}{11}$. Compute $\dfrac{2 + 3 + \cdots + 2n}{4 + 5 + \cdots + 4n}$.

**Problem 8.** The quadratic polynomial $f(x)$ has a zero at $x = 2$. The polynomial $f(f(x))$ has only one real zero, at $x = 5$. Compute $f(0)$.

**Problem 9.** The Local Area Inspirational Math Exam comprises 15 questions. All answers are integers ranging from 000 to 999, inclusive. If the 15 answers form an arithmetic progression with the largest possible difference, compute the largest possible sum of those 15 answers.

**Problem 10.** Circle $\omega_1$ has center $O$, which is on circle $\omega_2$. The circles intersect at points $A$ and $C$. Point $B$ lies on $\omega_2$ such that $BA = 37$, $BO = 17$, and $BC = 7$. Compute the area of $\omega_1$.

## 2 Individual Answers

**Answer 1.** 6455

**Answer 2.** $\frac{12}{5}$

**Answer 3.** $\frac{4}{17}$

**Answer 4.** 40

**Answer 5.** 69

**Answer 6.** 12

**Answer 7.** $\frac{27}{106}$

**Answer 8.** $-\frac{32}{9}$

**Answer 9.** 7530

**Answer 10.** $548\pi$

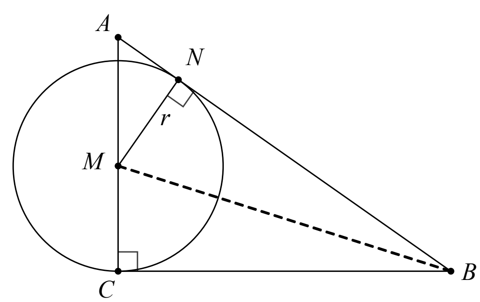
# 3 Individual Solutions

**Problem 1.** Compute the $2011^{\text{th}}$ smallest positive integer $N$ that gains an extra digit when doubled.

**Solution 1.** Let $S$ be the set of numbers that gain an extra digit when doubled. First notice that the numbers in $S$ are precisely those whose first digit is at least 5. Thus there are five one-digit numbers in $S$, 50 two-digit numbers in $S$, and 500 three-digit numbers in $S$. Therefore 5000 is the $556^{\text{th}}$ smallest number in $S$, and because all four-digit numbers greater than 5000 are in $S$, the $2011^{\text{th}}$ smallest number in $S$ is $5000+(2011-556) = \mathbf{6455}$.

**Problem 2.** In triangle $ABC$, $C$ is a right angle and $M$ is on $\overline{AC}$. A circle with radius $r$ is centered at $M$, is tangent to $\overline{AB}$, and is tangent to $\overline{BC}$ at $C$. If $AC = 5$ and $BC = 12$, compute $r$.

**Solution 2.** Let $N$ be the point of tangency of the circle with $\overline{AB}$ and draw $\overline{MB}$, as shown below.




Because $\triangle BMC$ and $\triangle BMN$ are right triangles sharing a hypotenuse, and $\overline{MN}$ and $\overline{MC}$ are radii, $\triangle BMC \cong \triangle BMN$. Thus $BN = 12$ and $AN = 1$. Also $\triangle ANM \sim \triangle ACB$ because the right triangles share $\angle A$, so $\frac{NM}{AN} = \frac{CB}{AC}$. Therefore $\frac{r}{1} = \frac{12}{5}$, so $r = \mathbf{\frac{12}{5}}$.

**Problem 3.** The product of the first five terms of a geometric progression is 32. If the fourth term is 17, compute the second term.

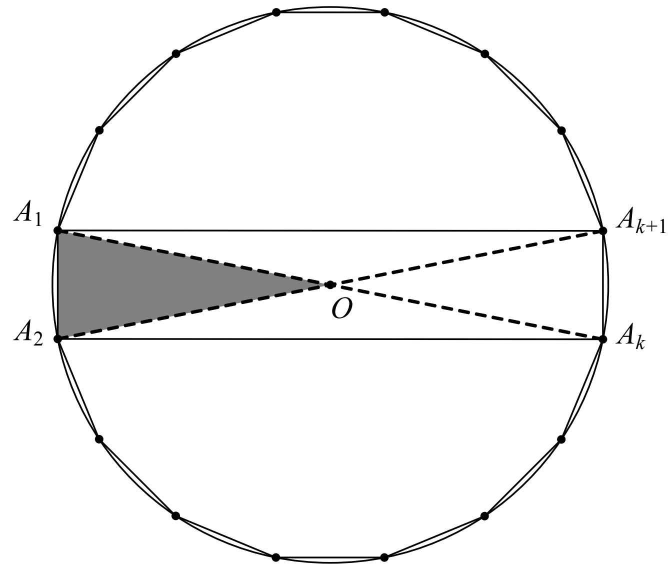
**Solution 3.** Let $a$ be the third term of the geometric progression, and let $r$ be the common ratio. Then the product of the first five terms is

$$(ar^{-2})(ar^{-1})(a)(ar)(ar^2) = a^5 = 32,$$

so $a = 2$. Because the fourth term is 17, $r = \frac{17}{a} = \frac{17}{2}$. The second term is $ar^{-1} = \frac{2}{17/2} = \mathbf{\frac{4}{17}}$.

**Problem 4.** Polygon $A_1A_2\ldots A_n$ is a regular $n$-gon. For some integer $k < n$, quadrilateral $A_1A_2A_kA_{k+1}$ is a rectangle of area 6. If the area of $A_1A_2\ldots A_n$ is 60, compute $n$.

**Solution 4.** Because $A_1A_2A_kA_{k+1}$ is a rectangle, $n$ must be even, and moreover, $k = \frac{n}{2}$. Also, the rectangle's diagonals meet at the center $O$ of the circumbscribing circle. $O$ is also the center of the $n$-gon. The diagram below shows the case $n = 16$.




Then $[A_1A_2O] = \frac{1}{4}[A_1A_2A_kA_{k+1}] = \frac{1}{n}[A_1A_2\ldots A_n] = 60$. So $\frac{1}{4}(6) = \frac{1}{n}(60)$, and $n = \mathbf{40}$.

**Problem 5.** A bag contains 20 lavender marbles, 12 emerald marbles, and some number of orange marbles. If the probability of drawing an orange marble in one try is $\frac{1}{y}$, compute the sum of all possible integer values of $y$.

**Solution 5.** Let $x$ be the number of orange marbles. Then the probability of drawing an orange marble is $\frac{x}{x+20+12} = \frac{x}{x+32}$. If this probability equals $\frac{1}{y}$, then $y = \frac{x+32}{x} = 1 + \frac{32}{x}$. This expression represents an integer only when $x$ is a factor of 32, thus $x \in \{1, 2, 4, 8, 16, 32\}$. The corresponding $y$-values are $33, 17, 9, 5, 3$, and $2$, and their sum is $\mathbf{69}$.

**Problem 6.** Compute the number of ordered quadruples of integers $(a, b, c, d)$ satisfying the following system of equations:

$$\begin{cases} abc &= 12{,}000 \\ bcd &= 24{,}000 \\ cda &= 36{,}000. \end{cases}$$

**Solution 6.** From the first two equations, conclude that $d = 2a$. From the last two, $3b = 2a$. Thus all solutions to the system will be of the form $(3K, 2K, c, 6K)$ for some integer $K$. Substituting these expressions into the system, each equation now becomes $cK^2 = 2000 = 2^4 \cdot 5^3$. So $K^2$ is of the form $2^{2m}5^{2n}$. There are 3 choices for $m$ and 2 for $n$, so there are 6 values for $K^2$, which means there are **12** solutions overall, including negative values for $K$.

Although the problem does not require finding them, the twelve values of $K$ are $\pm1$, $\pm2$, $\pm4$, $\pm5$, $\pm10$, $\pm20$. These values yield the following quadruples $(a, b, c, d)$:

$$\begin{gathered}(3, 2, 2000, 6), (-3, -2, 2000, -6),\\ (6, 4, 500, 12), (-6, -4, 500, -12),\\ (12, 8, 125, 24), (-12, -8, 125, -24),\\ (15, 10, 80, 30), (-15, -10, 80, -30),\\ (30, 20, 20, 60), (-30, -20, 20, -60),\\ (60, 40, 5, 120), (-60, -40, 5, -120).\end{gathered}$$

**Problem 7.** Let $n$ be a positive integer such that $\dfrac{3+4+\cdots+3n}{5+6+\cdots+5n}=\dfrac{4}{11}$. Compute $\dfrac{2+3+\cdots+2n}{4+5+\cdots+4n}$.

**Solution 7.** In simplifying the numerator and denominator of the left side of the equation, notice that

$$\begin{aligned} k+(k+1)+\cdots+kn &= \frac{1}{2}(kn(kn+1)-k(k-1)) \\ &= \frac{1}{2}(k(n+1)(kn-k+1)). \end{aligned}$$

This identity allows the given equation to be transformed:

$$\begin{aligned} \frac{3(n+1)(3n-3+1)}{5(n+1)(5n-5+1)} &= \frac{4}{11} \\ \frac{3(n+1)(3n-2)}{5(n+1)(5n-4)} &= \frac{4}{11} \\ \frac{3n-2}{5n-4} &= \frac{20}{33}. \end{aligned}$$

Solving this last equation yields $n=14$. Using the same identity twice more, for $n=14$ and $k=2$ and $k=4$, the desired quantity is $\frac{2(2n-1)}{4(4n-3)}=\boldsymbol{\frac{27}{106}}$.

**Problem 8.** The quadratic polynomial $f(x)$ has a zero at $x=2$. The polynomial $f(f(x))$ has only one real zero, at $x=5$. Compute $f(0)$.

**Solution 8.** Let $f(x)=a(x-b)^2+c$. The graph of $f$ is symmetric about $x=b$, so the graph of $y=f(f(x))$ is also symmetric about $x=b$. If $b\neq 5$, then $2b-5$, the reflection of 5 across $b$, must be a zero of $f(f(x))$. Because $f(f(x))$ has exactly one zero, $b=5$.

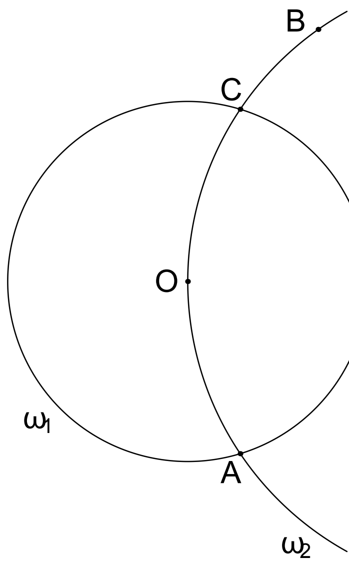
Because $f(2)=0$ and $f$ is symmetric about $x=5$, the other zero of $f$ is $x=8$. Because the zeros of $f$ are at 2 and 8 and $f(5)$ is a zero of $f$, either $f(5)=2$ or $f(5)=8$. The following argument shows that $f(5)=8$ is impossible. Because $f$ is continuous, if $f(5)=8$, then $f(x_0)=2$ for some $x_0$ in the interval $2<x_0<5$. In that case, $f(f(x_0))=0$, so 5 would not be a unique zero of $f(f(x))$. Therefore $f(5)=2$ and $c=2$. Setting $f(2)=0$ yields the equation $a(2-5)^2+2=0$, so $a=-\frac{2}{9}$, and $f(0)=\boldsymbol{-\frac{32}{9}}$.

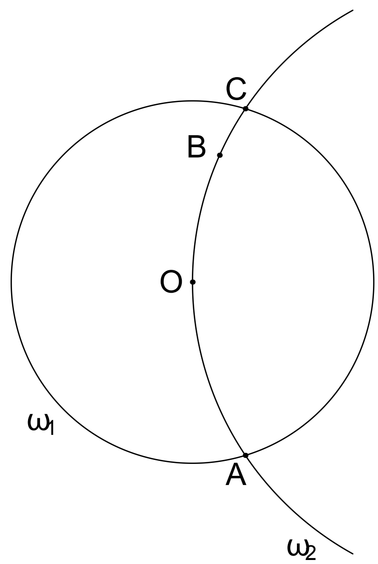
**Problem 9.** The Local Area Inspirational Math Exam comprises 15 questions. All answers are integers ranging from 000 to 999, inclusive. If the 15 answers form an arithmetic progression with the largest possible difference, compute the largest possible sum of those 15 answers.

**Solution 9.** Let $a$ represent the middle ($8^{\text{th}}$) term of the sequence, and let $d$ be the difference. Then the terms of the sequence are $a-7d, a-6d, \ldots, a+6d, a+7d$, their sum is $15a$, and the difference between the largest and the smallest terms is $14d$. The largest $d$ such that $14d\leq 999$ is $d=71$. Thus the largest possible value for $a$ is $999-7\cdot 71=502$. The maximal sum of the sequence is therefore $15a=\mathbf{7530}$.

**Problem 10.** Circle $\omega_1$ has center $O$, which is on circle $\omega_2$. The circles intersect at points $A$ and $C$. Point $B$ lies on $\omega_2$ such that $BA=37$, $BO=17$, and $BC=7$. Compute the area of $\omega_1$.

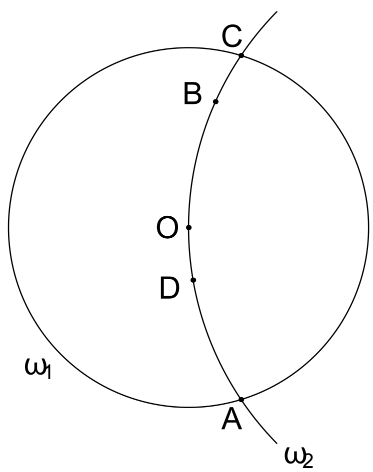
**Solution 10.** The points $O, A, B, C$ all lie on $\omega_2$ in some order. There are two possible cases to consider: either $B$ is outside circle $\omega_1$, or it is inside the circle, as shown below.



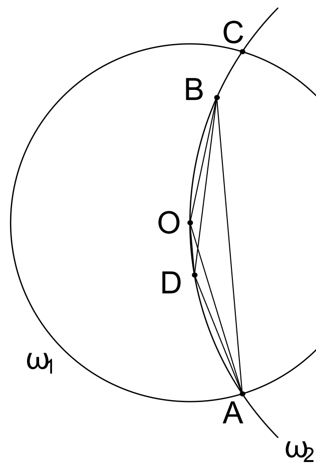




The following argument shows that the first case is impossible. By the Triangle Inequality on $\triangle ABO$, the radius $r_1$ of circle $\omega_1$ must be at least 20. But because $B$ is outside $\omega_1$, $BO > r_1$, which is impossible, because $BO = 17$. So $B$ must be inside the circle.

Construct point $D$ on minor arc $AO$ of circle $\omega_2$, so that $AD = OB$ (and therefore $DO = BC$).




Because $A, D, O, B$ all lie on $\omega_2$, Ptolemy's Theorem applies to quadrilateral $ADOB$.




Therefore $AD \cdot OB + OD \cdot AB = AO \cdot DB = r_1^2$. Substituting $AD = OB = 17$, $DO = BC = 7$, and $AB = 37$ yields $r_1^2 = 37 \cdot 7 + 17^2 = 548$. Thus the area of $\omega_1$ is $\mathbf{548\pi}$.

## 4 Team Problems

**Problem 1.** If $1, x, y$ is a geometric sequence and $x, y, 3$ is an arithmetic sequence, compute the maximum value of $x + y$.

**Problem 2.** Define the sequence of positive integers $\{a_n\}$ as follows:

$$\begin{cases} a_1 = 1; \\ \text{for } n \geq 2,\ a_n \text{ is the smallest possible positive value of } n - {a_k}^2, \text{ for } 1 \leq k < n. \end{cases}$$

For example, $a_2 = 2 - 1^2 = 1$, and $a_3 = 3 - 1^2 = 2$. Compute $a_1 + a_2 + \cdots + a_{50}$.

**Problem 3.** Compute the base $b$ for which $253_b \cdot 341_b = \underline{7}\,\underline{4}\,\underline{X}\,\underline{Y}\,\underline{Z}_{\,b}$, for some base-$b$ digits $X, Y, Z$.

**Problem 4.** Some portions of the line $y = 4x$ lie below the curve $y = 10\pi \sin^2 x$, and other portions lie above the curve. Compute the sum of the lengths of all the segments of the graph of $y = 4x$ that lie in the first quadrant, below the graph of $y = 10\pi \sin^2 x$.

**Problem 5.** In equilateral hexagon $ABCDEF$, $\text{m}\angle A = 2\text{m}\angle C = 2\text{m}\angle E = 5\text{m}\angle D = 10\text{m}\angle B = 10\text{m}\angle F$, and diagonal $BE = 3$. Compute $[ABCDEF]$, that is, the area of $ABCDEF$.

**Problem 6.** The *taxicab distance* between points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ is defined as $d(A, B) = |x_A - x_B| + |y_A - y_B|$. Given some $s > 0$ and points $A = (x_A, y_A)$ and $B = (x_B, y_B)$, define the *taxicab ellipse* with foci $A = (x_A, y_A)$ and $B = (x_B, y_B)$ to be the set of points $\{Q \mid d(A, Q) + d(B, Q) = s\}$. Compute the area enclosed by the taxicab ellipse with foci $(0, 5)$ and $(12, 0)$, passing through $(1, -1)$.

**Problem 7.** The function $f$ satisfies the relation $f(n) = f(n-1)f(n-2)$ for all integers $n$, and $f(n) > 0$ for all positive integers $n$. If $f(1) = \frac{f(2)}{512}$ and $\frac{1}{f(1)} = 2f(2)$, compute $f(f(4))$.

**Problem 8.** Frank Narf accidentally read a degree $n$ polynomial with integer coefficients backwards. That is, he read $a_n x^n + \ldots + a_1 x + a_0$ as $a_0 x^n + \ldots + a_{n-1} x + a_n$. Luckily, the reversed polynomial had the same zeros as the original polynomial. All the reversed polynomial's zeros were real, and also integers. If $1 \leq n \leq 7$, compute the number of such polynomials such that $\text{GCD}(a_0, a_1, \ldots, a_n) = 1$.

**Problem 9.** Given a regular 16-gon, extend three of its sides to form a triangle none of whose vertices lie on the 16-gon itself. Compute the number of noncongruent triangles that can be formed in this manner.

**Problem 10.** Two square tiles of area 9 are placed with one directly on top of the other. The top tile is then rotated about its center by an acute angle $\theta$. If the area of the overlapping region is 8, compute $\sin\theta + \cos\theta$.

## 5 Team Answers

**Answer 1.** $\frac{15}{4}$

**Answer 2.** 253

**Answer 3.** 20

**Answer 4.** $\frac{5\pi}{4}\sqrt{17}$

**Answer 5.** $\frac{9}{2}$

**Answer 6.** 96

**Answer 7.** 4096

**Answer 8.** 70

**Answer 9.** 11

**Answer 10.** $\frac{5}{4}$

## 6 Team Solutions

**Problem 1.** If $1, x, y$ is a geometric sequence and $x, y, 3$ is an arithmetic sequence, compute the maximum value of $x + y$.

**Solution 1.** The common ratio in the geometric sequence $1, x, y$ is $\frac{x}{1} = x$, so $y = x^2$. The arithmetic sequence $x, y, 3$ has a common difference, so $y - x = 3 - y$. Substituting $y = x^2$ in the equation yields

$$\begin{aligned} x^2 - x &= 3 - x^2 \\ 2x^2 - x - 3 &= 0, \end{aligned}$$

from which $x = \frac{3}{2}$ or $-1$. The respective values of $y$ are $y = x^2 = \frac{9}{4}$ or 1. Thus the possible values of $x + y$ are $\frac{15}{4}$ and 0, so the answer is $\mathbf{\frac{15}{4}}$.

**Problem 2.** Define the sequence of positive integers $\{a_n\}$ as follows:

$$\begin{cases} a_1 = 1; \\ \text{for } n \geq 2,\ a_n \text{ is the smallest possible positive value of } n - {a_k}^2, \text{ for } 1 \leq k < n. \end{cases}$$

For example, $a_2 = 2 - 1^2 = 1$, and $a_3 = 3 - 1^2 = 2$. Compute $a_1 + a_2 + \cdots + a_{50}$.

**Solution 2.** The requirement that $a_n$ be the smallest positive value of $n - {a_k}^2$ for $k < n$ is equivalent to determining the largest value of $a_k$ such that ${a_k}^2 < n$. For $n = 3$, use either $a_1 = a_2 = 1$ to find $a_3 = 3 - 1^2 = 2$. For $n = 4$, the strict inequality eliminates $a_3$, so $a_4 = 4 - 1^2 = 3$, but $a_3$ can be used to compute $a_5 = 5 - 2^2 = 1$. In fact, until $n = 10$, the largest allowable prior value of $a_k$ is $a_3 = 2$, yielding the values $a_6 = 2$, $a_7 = 3$, $a_8 = 4$, $a_9 = 5$. In general, this pattern continues: from $n = m^2 + 1$ until $n = (m + 1)^2$, the values of $a_n$ increase from 1 to $2m + 1$.

Let $S_m = 1 + 2 + \cdots + (2m + 1)$. Then the problem reduces to computing $S_0 + S_1 + \cdots + S_6 + 1$, because $a_{49} = 49 - 6^2$ while $a_{50} = 50 - 7^2 = 1$. $S_m = \frac{(2m+1)(2m+2)}{2} = 2m^2 + 3m + 1$, so

$$\begin{aligned} S_0 + S_1 + S_2 + S_3 + S_4 + S_5 + S_6 &= 1 + 6 + 15 + 28 + 45 + 66 + 91 \\ &= 252. \end{aligned}$$

Therefore the desired sum is $252 + 1 = \mathbf{253}$.

**Problem 3.** Compute the base $b$ for which $253_b \cdot 341_b = \underline{7}\,\underline{4}\,\underline{X}\,\underline{Y}\,\underline{Z}\,_b$, for some base-$b$ digits $X, Y, Z$.

**Solution 3.** Write $253_b \cdot 341_b = (2b^2 + 5b + 3)(3b^2 + 4b + 1) = 6b^4 + 23b^3 + 31b^2 + 17b + 3$. Compare the coefficients in this polynomial to the digits in the numeral $\underline{7}\ \underline{4}\ \underline{X}\ \underline{Y}\ \underline{Z}$. In the polynomial, the coefficient of $b^4$ is 6, so there must be a carry from the $b^3$ place to get the $7b^4$ in the numeral. After the carry, there should be no more than 4 left for the coefficient of $b^3$ as only one $b$ is carried. Therefore $23 - b \leq 4$ or $b \geq 19$.

By comparing digits, note that $Z = 3$. Then

$$\begin{aligned} 6b^4 + 23b^3 + 31b^2 + 17b &= \underline{7}\ \underline{4}\ \underline{X}\ \underline{Y}\ \underline{0} \\ &= 7b^4 + 4b^3 + X \cdot b^2 + Y \cdot b. \end{aligned}$$

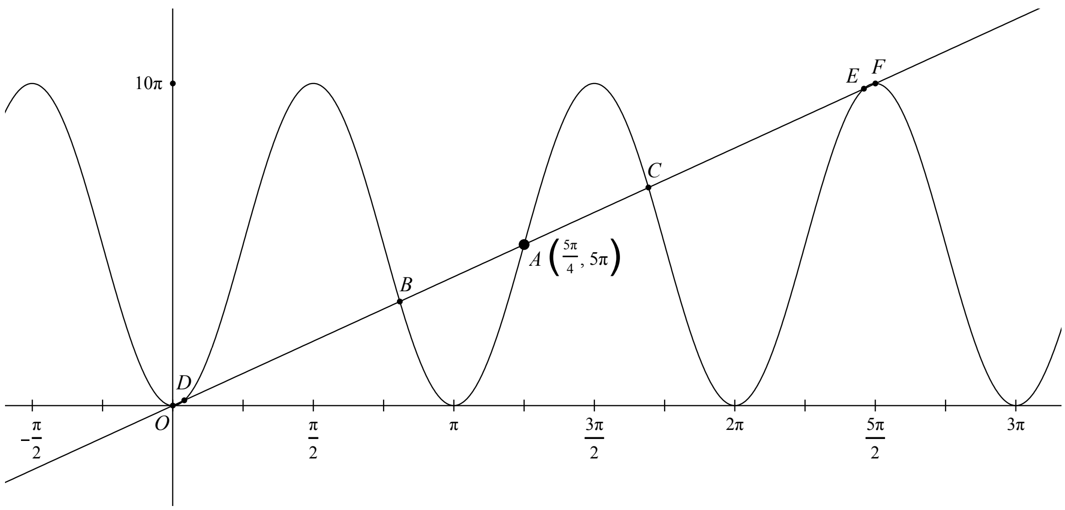
Because $b > 0$, this equation can be simplified to

$$b^3 + X \cdot b + Y = 19b^2 + 31b + 17.$$

Thus $Y = 17$ and $b^2 + X = 19b + 31$, from which $b(b - 19) = 31 - X$. The expression on the left side is positive (because $b > 19$) and the expression on the right side is at most 31 (because $X > 0$), so the only possible solution is $b = 20$, $X = 11$. The answer is **20**.

**Problem 4.** Some portions of the line $y = 4x$ lie below the curve $y = 10\pi \sin^2 x$, and other portions lie above the curve. Compute the sum of the lengths of all the segments of the graph of $y = 4x$ that lie in the first quadrant, below the graph of $y = 10\pi \sin^2 x$.

**Solution 4.** Notice first that all intersections of the two graphs occur in the interval $0 \le x \le \frac{5\pi}{2}$, because the maximum value of $10\pi \sin^2 x$ is $10\pi$ (at odd multiples of $\frac{\pi}{2}$), and $4x > 10\pi$ when $x > \frac{5\pi}{2}$. The graphs are shown below.




Within that interval, both graphs are symmetric about the point $A = \left(\frac{5\pi}{4}, 5\pi\right)$. For the case of $y = 10\pi \sin^2 x$, this symmetry can be seen by using the power-reducing identity $\sin^2 x = \frac{1-\cos 2x}{2}$. Then the equation becomes $y = 5\pi - 5\pi \cos 2x$, which has amplitude $5\pi$ about the line $y = 5\pi$, and which crosses the line $y = 5\pi$ for $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$.

Label the points of intersection $A, B, C, D, E, F$, and $O$ as shown. Then $\overline{AB} \cong \overline{AC}$, $\overline{BD} \cong \overline{CE}$, and $\overline{OD} \cong \overline{EF}$. Thus

$$\begin{aligned} BD + AC + EF &= OD + DB + BA \\ &= OA. \end{aligned}$$

By the Pythagorean Theorem,

$$\begin{aligned} OA &= \sqrt{\left(\frac{5\pi}{4}\right)^2 + (5\pi)^2} \\ &= \frac{5\pi}{4}\sqrt{1^2 + 4^2} \\ &= \mathbf{\frac{5\pi}{4}\sqrt{17}}. \end{aligned}$$

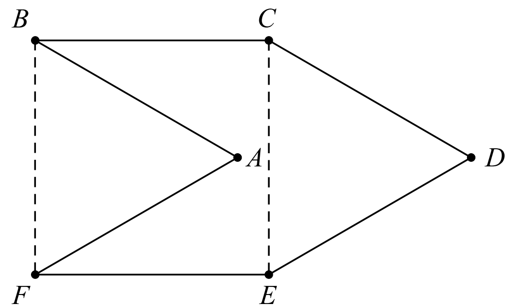
**Problem 5.** In equilateral hexagon $ABCDEF$, $m\angle A = 2m\angle C = 2m\angle E = 5m\angle D = 10m\angle B = 10m\angle F$, and diagonal $BE = 3$. Compute $[ABCDEF]$, that is, the area of $ABCDEF$.

**Solution 5.** Let $m\angle B = \alpha$. Then the sum of the measures of the angles in the hexagon is:

$$\begin{aligned} 720^\circ &= m\angle A + m\angle C + m\angle E + m\angle D + m\angle B + m\angle F \\ &= 10\alpha + 5\alpha + 5\alpha + 2\alpha + \alpha + \alpha = 24\alpha. \end{aligned}$$

Thus $30^\circ = \alpha$ and $m\angle A = 300^\circ$, so the exterior angle at $A$ has measure $60^\circ = m\angle D$. Further, because $AB = CD$ and $DE = AF$, it follows that $\triangle CDE \cong \triangle BAF$. Thus

$$[ABCDEF] = [ABCEF] + [CDE] = [ABCEF] + [ABF] = [BCEF].$$




To compute $[BCEF]$, notice that because $m\angle D = 60^\circ$, $\triangle CDE$ is equilateral. In addition,

$$\begin{aligned} 150^\circ &= m\angle BCD \\ &= m\angle BCE + m\angle DCE = m\angle BCE + 60^\circ. \end{aligned}$$

Therefore $m\angle BCE = 90^\circ$. Similarly, because the hexagon is symmetric, $m\angle CEF = 90^\circ$, so quadrilateral $BCEF$ is actually a square with side length 3. Thus $CE = \frac{BE}{\sqrt{2}} = \frac{3}{\sqrt{2}}$, and $[ABCDEF] = [BCEF] = \mathbf{\frac{9}{2}}$.

**Alternate Solution:** Calculate the angles of the hexagon as in the first solution. Then proceed as follows.

First, $ABCDEF$ can be partitioned into four congruent triangles. Because the hexagon is equilateral and $m\angle ABC = m\angle AFE = 30^\circ$, it follows that $\triangle ABC$ and $\triangle AFE$ are congruent isosceles triangles whose base angles measure $75^\circ$. Next, $m\angle ABC + m\angle BCD = 30^\circ + 150^\circ = 180^\circ$, so $\overline{AB} \parallel \overline{CD}$. Because these two segments are also congruent, quadrilateral $ABCD$ is a parallelogram. In particular, $\triangle CDA \cong \triangle ABC$. Similarly, $\triangle EDA \cong \triangle AFE$.

Now let $a = AC = AE$ be the length of the base of these isosceles triangles, and let $b = AB$ be the length of the other sides (or of the equilateral hexagon). Because the four triangles are congruent, $[ABCDEF] = [ABC] + [ACD] + [ADE] + [AEF] = 4[ABC] = 4 \cdot \frac{1}{2} b^2 \sin 30^\circ = b^2$.

Applying the Law of Cosines to $\triangle ABC$ gives $a^2 = b^2 + b^2 - 2b^2 \cos 30^\circ = (2 - \sqrt{3})b^2$. Because $4 - 2\sqrt{3} = (\sqrt{3} - 1)^2$, this gives $a = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right) b$. Using the given length $BE = 3$ and applying the Law of Cosines to $\triangle ABE$ gives

$$\begin{aligned} 9 &= a^2 + b^2 - 2ab\cos 135^\circ \\ &= a^2 + b^2 + \sqrt{2}ab \\ &= (2 - \sqrt{3})b^2 + b^2 + (\sqrt{3} - 1)b^2 \\ &= 2b^2. \end{aligned}$$

Thus $[ABCDEF] = b^2 = \mathbf{\frac{9}{2}}$.

**Problem 6.** The *taxicab distance* between points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ is defined as $d(A, B) = |x_A - x_B| + |y_A - y_B|$. Given some $s > 0$ and points $A = (x_A, y_A)$ and $B = (x_B, y_B)$, define the *taxicab ellipse* with foci $A = (x_A, y_A)$ and $B = (x_B, y_B)$ to be the set of points $\{Q \mid d(A, Q) + d(B, Q) = s\}$. Compute the area enclosed by the taxicab ellipse with foci $(0, 5)$ and $(12, 0)$, passing through $(1, -1)$.

**Solution 6.** Let $A = (0, 5)$ and $B = (12, 0)$, and let $C = (1, -1)$. First compute the distance sum: $d(A, C) + d(B, C) = 19$. Notice that if $P = (x, y)$ is on the segment from $(0, -1)$ to $(12, -1)$, then $d(A, P) + d(B, P)$ is constant. This is because if $0 < x < 12$,

$$\begin{aligned} d(A,P) + d(B,P) &= |0 - x| + |5 - (-1)| + |12 - x| + |0 - (-1)| \\ &= x + 6 + (12 - x) + 1 \\ &= 19. \end{aligned}$$

Similarly, $d(A, P) + d(P, B) = 19$ whenever $P$ is on the segment from $(0, 6)$ to $(12, 6)$. If $P$ is on the segment from $(13, 0)$ to $(13, 5)$, then $P$'s coordinates are $(13, y)$, with $0 \le y \le 5$, and thus

$$\begin{aligned} d(A,P) + d(B,P) &= |0 - 13| + |5 - y| + |12 - 13| + |0 - y| \\ &= 13 + (5 - y) + 1 + y \\ &= 19. \end{aligned}$$

Similarly, $d(A, P) + d(P, B) = 19$ whenever $P$ is on the segment from $(-1, 0)$ to $(-1, 5)$.

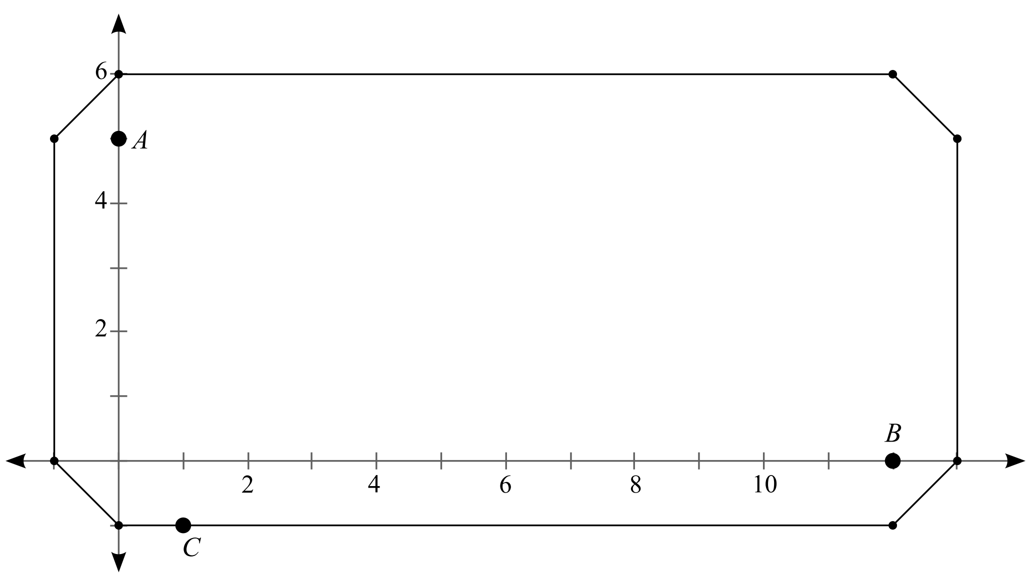
Finally, if $P$ is on the segment from $(12, -1)$ to $(13, 0)$, then $d(A, P) + d(B, P)$ is constant:

$$\begin{aligned} d(A,P) + d(B,P) &= |0 - x| + |5 - y| + |12 - x| + |0 - y| \\ &= x + (5 - y) + (x - 12) + (-y) \\ &= 2x - 2y - 7, \end{aligned}$$

and because the line segment has equation $x - y = 13$, this expression reduces to

$$\begin{aligned} d(A,P) + d(B,P) &= 2(x - y) - 7 \\ &= 2(13) - 7 \\ &= 19. \end{aligned}$$

Similarly, $d(A, P) + d(B, P) = 19$ on the segments joining $(13, 5)$ and $(12, 6)$, $(0, 6)$ and $(-1, 5)$, and $(-1, 0)$ to $(0, -1)$. The shape of the "ellipse" is given below.




The simplest way to compute the polygon's area is to subtract the areas of the four corner triangles from that of the enclosing rectangle. The enclosing rectangle's area is $14 \cdot 7 = 98$, while each triangle has area $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. Thus the area is $98 - 4 \cdot \frac{1}{2} = \mathbf{96}$.

**Problem 7.** The function $f$ satisfies the relation $f(n) = f(n-1)f(n-2)$ for all integers $n$, and $f(n) > 0$ for all positive integers $n$. If $f(1) = \frac{f(2)}{512}$ and $\frac{1}{f(1)} = 2f(2)$, compute $f(f(4))$.

**Solution 7.** Substituting yields $\frac{512}{f(2)} = 2f(2) \Rightarrow (f(2))^2 = 256 \Rightarrow f(2) = 16$. Therefore $f(1) = \frac{1}{32}$. Using the recursion, $f(3) = \frac{1}{2}$ and $f(4) = 8$. So $f(f(4)) = f(8)$. Continue to apply the recursion:

$$f(5) = 4, \quad f(6) = 32, \quad f(7) = 128, \quad f(8) = \mathbf{4096}.$$

**Alternate Solution:** Let $g(n) = \log_2 f(n)$. Then $g(n) = g(n-1) + g(n-2)$, with initial conditions $g(1) = g(2) - 9$ and $-g(1) = 1 + g(2)$. From this, $g(1) = -5$ and $g(2) = 4$, and from the recursion,

$$g(3) = -1, \quad g(4) = 3,$$

so $f(4) = 2^{g(4)} = 8$. Continue to apply the recursion:

$$g(5) = 2, \quad g(6) = 5, \quad g(7) = 7, \quad g(8) = 12.$$

Because $g(f(4)) = 12$, it follows that $f(f(4)) = 2^{12} = \mathbf{4096}$.

**Problem 8.** Frank Narf accidentally read a degree $n$ polynomial with integer coefficients backwards. That is, he read $a_n x^n + \ldots + a_1 x + a_0$ as $a_0 x^n + \ldots + a_{n-1}x + a_n$. Luckily, the reversed polynomial had the same zeros as the original polynomial. All the reversed polynomial's zeros were real, and also integers. If $1 \le n \le 7$, compute the number of such polynomials such that $\text{GCD}(a_0, a_1, \ldots, a_n) = 1$.

**Solution 8.** When the coefficients of a polynomial $f$ are reversed to form a new polynomial $g$, the zeros of $g$ are the reciprocals of the zeros of $f$: $r$ is a zero of $f$ if and only if $r^{-1}$ is a zero of $g$. In this case, the two polynomials have the *same* zeros; that is, whenever $r$ is a zero of either, so must be $r^{-1}$. Furthermore, both $r$ and $r^{-1}$ must be real as well as integers, so $r = \pm 1$. As the only zeros are $\pm 1$, and the greatest common divisor of all the coefficients is 1, the polynomial must have leading coefficient 1 or $-1$.

Thus

$$\begin{aligned} f(x) &= \pm(x \pm 1)(x \pm 1) \cdots (x \pm 1) \\ &= \pm(x+1)^k (x-1)^{n-k}. \end{aligned}$$

If $A_n$ is the number of such degree $n$ polynomials, then there are $n+1$ choices for $k$, $0 \le k \le n$. Thus $A_n = 2(n+1)$.

The number of such degree $n$ polynomials for $1 \le n \le 7$ is the sum:

$$A_1 + A_2 + \ldots + A_7 = 2(2 + 3 + \ldots + 8) = 2 \cdot 35 = \mathbf{70}.$$

**Problem 9.** Given a regular 16-gon, extend three of its sides to form a triangle none of whose vertices lie on the 16-gon itself. Compute the number of noncongruent triangles that can be formed in this manner.

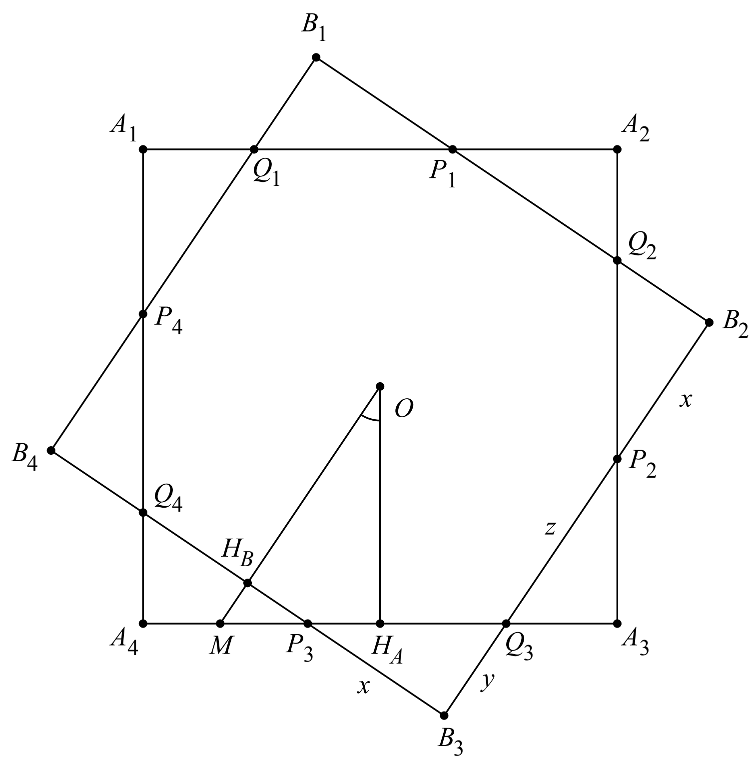
**Solution 9.** Label the sides of the polygon, in order, $s_0, s_1, \ldots, s_{15}$. First note that two sides of the polygon intersect at a vertex if and only if the sides are adjacent. So the sides chosen must be nonconsecutive. Second, if nonparallel sides $s_i$ and $s_j$ are extended, the angle of intersection is determined by $|i-j|$, as are the lengths of the extended portions of the segments. In other words, the *spacing* of the extended sides completely determines the shape of the triangle. So the problem reduces to selecting appropriate spacings, that is, finding integers $a, b, c \ge 2$ whose sum is 16. However, diametrically opposite sides are parallel, so (for example) the sides $s_3$ and $s_{11}$ cannot both be used. Thus none of $a, b, c$ may equal 8. Taking $s_0$ as the first side, the second side would be $s_{0+a} = s_a$, and the third side would be $s_{a+b}$, with $c$ sides between $s_{a+b}$ and $s_0$. To eliminate reflections and rotations, specify additionally that $a \ge b \ge c$. The allowable partitions are in the table below.

| $a$ | $b$ | $c$ | triangle |
|---|---|---|---|
| 12 | 2 | 2 | $s_0 s_{12} s_{14}$ |
| 11 | 3 | 2 | $s_0 s_{11} s_{14}$ |
| 10 | 4 | 2 | $s_0 s_{10} s_{14}$ |
| 10 | 3 | 3 | $s_0 s_{10} s_{13}$ |
| 9 | 5 | 2 | $s_0 s_9 s_{14}$ |
| 9 | 4 | 3 | $s_0 s_9 s_{13}$ |
| 7 | 7 | 2 | $s_0 s_7 s_{14}$ |
| 7 | 6 | 3 | $s_0 s_7 s_{13}$ |
| 7 | 5 | 4 | $s_0 s_7 s_{12}$ |
| 6 | 6 | 4 | $s_0 s_6 s_{12}$ |
| 6 | 5 | 5 | $s_0 s_6 s_{11}$ |

Thus there are **11** distinct such triangles.

**Problem 10.** Two square tiles of area 9 are placed with one directly on top of the other. The top tile is then rotated about its center by an acute angle $\theta$. If the area of the overlapping region is 8, compute $\sin\theta + \cos\theta$.

**Solution 10.** In the diagram below, $O$ is the center of both squares $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Let $P_1, P_2, P_3, P_4$ and $Q_1, Q_2, Q_3, Q_4$ be the intersections of the sides of the squares as shown. Let $H_A$ be on $\overline{A_3A_4}$ so that $\angle A_3H_AO$ is right. Similarly, let $H_B$ be on $\overline{B_3B_4}$ such that $\angle B_3H_BO$ is right. Then the angle by which $B_1B_2B_3B_4$ was rotated is $\angle H_AOH_B$. Extend $\overline{OH_B}$ to meet $\overline{A_3A_4}$ at $M$.




Both $\triangle H_AOM$ and $\triangle H_BP_3M$ are right triangles sharing acute $\angle M$, so $\triangle H_AOM \sim \triangle H_BP_3M$. By an analogous argument, both triangles are similar to $\triangle B_3P_3Q_3$. Thus $\text{m}\angle Q_3P_3B_3 = \theta$.

Now let $B_3P_3 = x, B_3Q_3 = y$, and $P_3Q_3 = z$. By symmetry, notice that $B_3P_3 = B_2P_2$ and that $P_3Q_3 = P_2Q_3$. Thus

$$x + y + z = B_3Q_3 + Q_3P_2 + P_2B_2 = B_2B_3 = 3.$$

By the Pythagorean Theorem, $x^2 + y^2 = z^2$. Therefore

$$\begin{aligned} x + y &= 3 - z \\ x^2 + y^2 + 2xy &= 9 - 6z + z^2 \\ 2xy &= 9 - 6z. \end{aligned}$$

The value of $xy$ can be determined from the areas of the four triangles $\triangle B_i P_i Q_i$. By symmetry, these four triangles are congruent to each other. Their total area is the area not in both squares, i.e., $9 - 8 = 1$. Thus $\frac{xy}{2} = \frac{1}{4}$, so $2xy = 1$.

Applying this result to the above equation,

$$\begin{aligned} 1 &= 9 - 6z \\ z &= \frac{4}{3}. \end{aligned}$$

The desired quantity is $\sin\theta + \cos\theta = \frac{x}{z} + \frac{y}{z}$, and

$$\begin{aligned} \frac{x}{z} + \frac{y}{z} &= \frac{x+y+z}{z} - \frac{z}{z} \\ &= \frac{3}{z} - 1 \\ &= \mathbf{\frac{5}{4}}. \end{aligned}$$

## 7 Power Question 2011: Power of Triangles

**Instructions:** The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

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The arrangement of numbers known as Pascal's Triangle has fascinated mathematicians for centuries. In fact, about 700 years before Pascal, the Indian mathematician Halayudha wrote about it in his commentaries to a then-1000-year-old treatise on verse structure by the Indian poet and mathematician Pingala, who called it the *Meruprastāra*, or "Mountain of Gems". In this Power Question, we'll explore some properties of Pingala's/Pascal's Triangle ("PT") and its variants.

Unless otherwise specified, the **only** definition, notation, and formulas you may use for PT are the definition, notation, and formulas given below.

PT consists of an infinite number of rows, numbered from 0 onwards. The $n^{\text{th}}$ row contains $n+1$ numbers, identified as $\text{Pa}(n,k)$, where $0 \le k \le n$. For all $n$, define $\text{Pa}(n,0) = \text{Pa}(n,n) = 1$. Then for $n > 1$ and $1 \le k \le n-1$, define $\text{Pa}(n,k) = \text{Pa}(n-1,k-1) + \text{Pa}(n-1,k)$. It is convenient to define $\text{Pa}(n,k) = 0$ when $k < 0$ or $k > n$. We write the nonzero values of PT in the familiar pyramid shown below.

**1**
$\text{Pa}(0,0)$

**1** $\text{Pa}(1,0)$    **1** $\text{Pa}(1,1)$

**1** $\text{Pa}(2,0)$    **2** $\text{Pa}(2,1)$    **1** $\text{Pa}(2,2)$

**1** $\text{Pa}(3,0)$    **3** $\text{Pa}(3,1)$    **3** $\text{Pa}(3,2)$    **1** $\text{Pa}(3,3)$

As is well known, $\text{Pa}(n,k)$ gives the number of ways of choosing a committee of $k$ people from a set of $n$ people, so a simple formula for $\text{Pa}(n,k)$ is $\text{Pa}(n,k) = \frac{n!}{k!(n-k)!}$. You may use this formula or the recursive definition above throughout this Power Question.

**1a.** For $n = 1, 2, 3, 4$, and $k = 4$, find $\text{Pa}(n,n) + \text{Pa}(n+1,n) + \cdots + \text{Pa}(n+k,n)$. [2]

**1b.** If $\text{Pa}(n,n) + \text{Pa}(n+1,n) + \cdots + \text{Pa}(n+k,n) = \text{Pa}(m,j)$, find and justify formulas for $m$ and $j$ in terms of $n$ and $k$. [2]

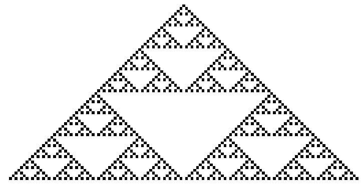
Consider the *parity* of each entry: define

$$\text{PaP}(n,k) = \begin{cases} 1 & \text{if } \text{Pa}(n,k) \text{ is odd,} \\ 0 & \text{if } \text{Pa}(n,k) \text{ is even.} \end{cases}$$

**2a.** Prove that $\text{PaP}(n,0) = \text{PaP}(n,n) = 1$ for all nonnegative integers $n$. [1]

**2b.** Compute rows $n = 0$ to $n = 8$ of PaP. [2]

You may have learned that the array of parities of Pascal's Triangle forms the famous fractal known as Sierpinski's Triangle. The first 128 rows are shown below, with a dot marking each entry where $\text{PaP}(n,k) = 1$.



The next problem helps explain this surprising connection.

**3a.** If $n = 2^j$ for some nonnegative integer $j$, and $0 < k < n$, show that $\mathrm{PaP}(n,k) = 0$. [2]

**3b.** Let $j \geq 0$, and suppose $n \geq 2^j$. Prove that $\mathrm{Pa}(n,k)$ has the same parity as the sum $\mathrm{Pa}(n-2^j, k-2^j) + \mathrm{Pa}(n-2^j, k)$, i.e., either both $\mathrm{Pa}(n,k)$ and the given sum are even, or both are odd. [2]

**3c.** If $j$ is an integer such that $2^j \leq n < 2^{j+1}$, and $k < 2^j$, prove that $\mathrm{PaP}(n,k) = \mathrm{PaP}(n-2^j, k)$. [2]

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**Clark's Triangle:** If the left side of PT is replaced with consecutive multiples of 6, starting with 0, but the right entries (except the first) and the generating rule are left unchanged, the result is called *Clark's Triangle.* If the $k^{\text{th}}$ entry of the $n^{\text{th}}$ row is denoted by $\mathrm{Cl}(n,k)$, then the formal rule is:

$$\begin{cases} \mathrm{Cl}(n,0) = 6n & \text{for all } n, \\ \mathrm{Cl}(n,n) = 1 & \text{for } n \geq 1, \\ \mathrm{Cl}(n,k) = \mathrm{Cl}(n-1,k-1) + \mathrm{Cl}(n-1,k) & \text{for } n \geq 1 \text{ and } 1 \leq k \leq n-1. \end{cases}$$

The first four rows of Clark's Triangle are given below.

**0**
$\mathrm{Cl}(0,0)$

**6** $\mathrm{Cl}(1,0)$ **1** $\mathrm{Cl}(1,1)$

**12** $\mathrm{Cl}(2,0)$ **7** $\mathrm{Cl}(2,1)$ **1** $\mathrm{Cl}(2,2)$

**18** $\mathrm{Cl}(3,0)$ **19** $\mathrm{Cl}(3,1)$ **8** $\mathrm{Cl}(3,2)$ **1** $\mathrm{Cl}(3,3)$

**4a.** Compute the next three rows of Clark's Triangle. [2]

**4b.** If $\mathrm{Cl}(n,1) = an^2 + bn + c$, determine the values of $a, b$, and $c$. [2]

**4c.** Prove the formula you found in 4b. [2]

**5a.** Compute $\mathrm{Cl}(11,2)$. [1]

**5b.** Find and justify a formula for $\mathrm{Cl}(n,2)$ in terms of $n$. [2]

**5c.** Compute $\mathrm{Cl}(11,3)$. [1]

**5d.** Find and justify a formula for $\mathrm{Cl}(n,3)$ in terms of $n$. [2]

**6.** Find and prove a closed formula (that is, a formula with a fixed number of terms and no "...") for $\mathrm{Cl}(n,k)$ in terms of $n$, $k$, and the Pa function. [3]

**Leibniz's Harmonic Triangle:** Consider the triangle formed by the rule

$$\begin{cases} \mathrm{Le}(n,0) = \frac{1}{n+1} & \text{for all } n, \\ \mathrm{Le}(n,n) = \frac{1}{n+1} & \text{for all } n, \\ \mathrm{Le}(n,k) = \mathrm{Le}(n+1,k) + \mathrm{Le}(n+1,k+1), & \text{for all } n \text{ and } 0 \le k \le n. \end{cases}$$

This triangle, discovered first by Leibniz, consists of reciprocals of integers as shown below.

$$\begin{array}{ccccccc} & & & \mathbf{1} & & & \\ & & & \mathrm{Le}(0,0) & & & \\ & & \mathbf{\frac{1}{2}} & & \mathbf{\frac{1}{2}} & & \\ & & \mathrm{Le}(1,0) & & \mathrm{Le}(1,1) & & \\ & \mathbf{\frac{1}{3}} & & \mathbf{\frac{1}{6}} & & \mathbf{\frac{1}{3}} & \\ & \mathrm{Le}(2,0) & & \mathrm{Le}(2,1) & & \mathrm{Le}(2,2) & \\ \mathbf{\frac{1}{4}} & & \mathbf{\frac{1}{12}} & & \mathbf{\frac{1}{12}} & & \mathbf{\frac{1}{4}} \\ \mathrm{Le}(3,0) & & \mathrm{Le}(3,1) & & \mathrm{Le}(3,2) & & \mathrm{Le}(3,3) \end{array}$$

For this contest, you may assume that $\mathrm{Le}(n,k) > 0$ whenever $0 \le k \le n$, and that $\mathrm{Le}(n,k)$ is undefined if $k < 0$ or $k > n$.

**7a.** Compute the entries in the next two rows of Leibniz's Triangle. [1]

**7b.** Compute $\mathrm{Le}(17,1)$. [1]

**7c.** Compute $\mathrm{Le}(17,2)$. [1]

**8a.** Find and justify a formula for $\mathrm{Le}(n,1)$ in terms of $n$. [2]

**8b.** Compute $\displaystyle\sum_{n=1}^{2011} \mathrm{Le}(n,1)$. [2]

**8c.** Find and justify a formula for $\mathrm{Le}(n,2)$ in terms of $n$. [2]

**9a.** If $\displaystyle\sum_{i=1}^{\infty} \mathrm{Le}(i,1) = \mathrm{Le}(n,k)$, determine the values of $n$ and $k$. [2]

**9b.** If $\displaystyle\sum_{i=m}^{\infty} \mathrm{Le}(i,m) = \mathrm{Le}(n,k)$, compute expressions for $n$ and $k$ in terms of $m$. [2]

**9c.** Justify your result in 9b. [2]

**10.** Find three distinct sets of positive integers $\{a,b,c,d\}$ with $a < b < c < d$ such that $\frac{1}{3} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$. [3]

**11.** Find and prove a closed formula (that is, a formula with a fixed number of terms and no "...") for $\mathrm{Le}(n,k)$ in terms of $n$, $k$, and the Pa function. [4]

# 8 Power Solutions

**1a.**

$$\begin{aligned}
\text{Pa}(1,1)+\text{Pa}(2,1)+\text{Pa}(3,1)+\text{Pa}(4,1)+\text{Pa}(5,1) &= 1+2+3+4+5 &= \mathbf{15}\\
\text{Pa}(2,2)+\text{Pa}(3,2)+\text{Pa}(4,2)+\text{Pa}(5,2)+\text{Pa}(6,2) &= 1+3+6+10+15 &= \mathbf{35}\\
\text{Pa}(3,3)+\text{Pa}(4,3)+\text{Pa}(5,3)+\text{Pa}(6,3)+\text{Pa}(7,3) &= 1+4+10+20+35 &= \mathbf{70}\\
\text{Pa}(4,4)+\text{Pa}(5,4)+\text{Pa}(6,4)+\text{Pa}(7,4)+\text{Pa}(8,4) &= 1+5+15+35+70 &= \mathbf{126}.
\end{aligned}$$

**1b.** Notice that $\text{Pa}(n,n)+\text{Pa}(n+1,n)+\cdots+\text{Pa}(n+k,n) = \text{Pa}(n+k+1,n+1)$, so $m = n+k+1$ and $j = n+1$. (By symmetry, $j = k$ is also correct.) The equation is true for all $n$ when $k = 0$, because the sum is simply $\text{Pa}(n,n)$ and the right side is $\text{Pa}(n+1,n+1)$, both of which are 1. Proceed by induction on $k$. If $\text{Pa}(n,n)+\text{Pa}(n+1,n)+\cdots+\text{Pa}(n+k,n) = \text{Pa}(n+k+1,n+1)$, then adding $\text{Pa}(n+k+1,n)$ to both sides yields $\text{Pa}(n+k+1,n)+\text{Pa}(n+k+1,n+1) = \text{Pa}(n+k+2,n+1)$ by the recursive rule for Pa.

**2a.** By definition of Pa, $\text{Pa}(n,0) = \text{Pa}(n,n) = 1$ for all nonnegative integers $n$, and this value is odd, so $\text{PaP}(n,0) = \text{PaP}(n,n) = 1$ by definition.

**2b.**

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&1&&0&&1&&&&&&\\
&&&&&1&&1&&1&&1&&&&&\\
&&&&1&&0&&0&&0&&1&&&&\\
&&&1&&1&&0&&0&&1&&1&&&\\
&&1&&0&&1&&0&&1&&0&&1&&\\
&1&&1&&1&&1&&1&&1&&1&&1&\\
1&&0&&0&&0&&0&&0&&0&&0&&1
\end{array}$$

**3a.** Notice that

$$\begin{aligned}
\text{Pa}(n,k) &= \frac{n!}{k!(n-k)!}\\
&= \frac{n}{k}\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}\\
&= \frac{n}{k}\cdot\text{Pa}(n-1,k-1).
\end{aligned}$$

Examining the right hand side in the case where $n = 2^j$ and $0 < k < n$, the second factor, $\text{Pa}(n-1,k-1)$ is an integer, and the first factor has an even numerator, so $\text{Pa}(n,k)$ is even. Therefore $\text{PaP}(n,k) = 0$.

**3b.** Proceed by induction on $j$. It is easier to follow the logic when the computations are expressed modulo 2. The claim is that $\text{Pa}(n,k) \equiv \text{Pa}(n-2^j,k-2^j)+\text{Pa}(n-2^j,k) \bmod 2$. If $j = 0$, so that $2^j = 1$, then this congruence is exactly the recursive definition of $\text{Pa}(n,k)$ when $0 < k < n$. For $k = 0$ and $k = n$, the left-hand side is 1 and the right-hand side is either $0+1$ or $1+0$. For other values of $k$, all three terms are zero.

Now, assume $\text{Pa}(n,k) \equiv \text{Pa}(n-2^j,k-2^j)+\text{Pa}(n-2^j,k) \bmod 2$ for some $j \geq 0$, and let $n \geq 2^{j+1}$. For any $k$, apply the inductive hypothesis three times:

$$\begin{aligned}
\text{Pa}(n,k) &\equiv \text{Pa}(n-2^j,k-2^j)+\text{Pa}(n-2^j,k) \bmod 2\\
&\equiv \text{Pa}(n-2^j-2^j,k-2^j-2^j)+\text{Pa}(n-2^j-2^j,k-2^j)\\
&\qquad +\text{Pa}(n-2^j-2^j,k-2^j)+\text{Pa}(n-2^j-2^j,k).
\end{aligned}$$

But $u+u \equiv 0 \bmod 2$ for any integer $u$. Thus the two occurrences of $\text{Pa}(n-2^j-2^j,k-2^j)$ cancel each other out, and

$$\begin{aligned}
\text{Pa}(n,k) &\equiv \text{Pa}(n-2^j-2^j,k-2^j-2^j)+\text{Pa}(n-2^j-2^j,k)\\
&\equiv \text{Pa}(n-2^{j+1},k-2^{j+1})+\text{Pa}(n-2^{j+1},k) \bmod 2.
\end{aligned}$$

**3c.** By part (b), $\text{Pa}(n,k) \equiv \text{Pa}(n-2^j, k-2^j) + \text{Pa}(n-2^j, k) \bmod 2$ when $j \geq 0$ and $n \geq 2^j$. If $2^j \leq n < 2^{j+1}$ and $0 \leq k < 2^j$, then $\text{Pa}(n-2^j, k-2^j) = 0$, so

$$\text{Pa}(n,k) \equiv \text{Pa}(n-2^j, k-2^j) + \text{Pa}(n-2^j, k) \equiv \text{Pa}(n-2^j, k) \bmod 2.$$

Thus $\text{PaP}(n,k) = \text{PaP}(n-2^j, k)$.

**Alternate Solution to 3c:**

Problem (3a) establishes the statement when $n = 2^j$. For $2^j < n < 2^{j+1}$, proceed by induction on $n$. Then $\text{PaP}(n,k) \equiv \text{PaP}(n-1,k-1) + \text{PaP}(n-1,k) \bmod 2$, while $\text{PaP}(n-1,k-1) = \text{PaP}(n-1-2^j, k-1)$ and $\text{PaP}(n-1,k) = \text{PaP}(n-1-2^j, k)$. But $\text{PaP}(n-1-2^j, k-1) + \text{PaP}(n-1-2^j, k) \equiv \text{PaP}(n-2^j, k) \bmod 2$, establishing the statement.

**4a.**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 24 | | 37 | | 27 | | 9 | | 1 | | |
| | 30 | | 61 | | 64 | | 36 | | 10 | | 1 | |
| 36 | | 91 | | 125 | | 100 | | 46 | | 11 | | 1 |

**4b.** Using the given values yields the system of equations below.

$$\begin{cases} \text{Cl}(1,1) &= 1 &= a(1)^2 + b(1) + c \\ \text{Cl}(2,1) &= 7 &= a(2)^2 + b(2) + c \\ \text{Cl}(3,1) &= 19 &= a(3)^2 + b(3) + c \end{cases}$$

Solving this system, $a = 3, b = -3, c = 1$.

**4c.** Use induction on $n$. For $n = 1, 2, 3$, the values above demonstrate the theorem. If $\text{Cl}(n,1) = 3n^2 - 3n + 1$, then $\text{Cl}(n+1,1) = \text{Cl}(n,0) + \text{Cl}(n,1) = 6n + (3n^2 - 3n + 1) = (3n^2 + 6n + 3) - (3n + 3) + 1 = 3(n+1)^2 - 3(n+1) + 1$.

**5a.** $\text{Cl}(11,2) = 1000$.

**5b.** $\text{Cl}(n,2) = (n-1)^3$. Use induction on $n$. First, rewrite $\text{Cl}(n,1) = 3n^2 - 3n + 1 = n^3 - (n-1)^3$, and notice that $\text{Cl}(2,2) = 1 = (2-1)^3$. Then if $\text{Cl}(n,2) = (n-1)^3$, using the recursive definition, $\text{Cl}(n+1,2) = \text{Cl}(n,2) + \text{Cl}(n,1) = (n-1)^3 + \left(n^3 - (n-1)^3\right) = n^3$.

**5c.** $\text{Cl}(11,3) = 2025$.

**5d.** Notice that $\text{Cl}(3,3) = 1 = 1^3$, and then for $n > 3$, $\text{Cl}(n,3) = \text{Cl}(n-1,2) + \text{Cl}(n-1,3)$; replacing $\text{Cl}(n-1,3)$ analogously on the right side yields the summation

$$\text{Cl}(n,3) = \text{Cl}(n-1,2) + \text{Cl}(n-2,2) + \ldots + \text{Cl}(3,2) + 1.$$

By 5b, $\text{Cl}(n-1,2) = (n-2)^3$, so this formula is equivalent to

$$\text{Cl}(n,3) = (n-2)^3 + (n-3)^3 + \ldots + 2^3 + 1.$$

Use the identity $1^3 + 2^3 + \cdots + m^3 = \frac{m^2(m+1)^2}{4}$ and substitute $n-2$ for $m$ to obtain $\text{Cl}(n,3) = \frac{(n-2)^2(n-1)^2}{4}$.

**6.** Notice that $\text{Cl}(n,0) = 6 \cdot \text{Pa}(n,1)$, and that for $n > 0$, $\text{Cl}(n,n) = \text{Pa}(n,n)$. From problem 4c, $\text{Cl}(n,1) = 6\text{Pa}(n,2) + 1$ (where $\text{Pa}(n,k) = 0$ if $k > n$).

For $k > 1$, repeated application of the formula $\text{Cl}(m,k) = \text{Cl}(m-1,k-1) + \text{Cl}(m-1,k)$ allows each value of $\text{Cl}(n,k)$ to be written as a sum:

$$\begin{aligned} \text{Cl}(n,k) &= \text{Cl}(n-1,k-1) + \text{Cl}(n-1,k) \\ &= \text{Cl}(n-1,k-1) + \text{Cl}(n-2,k-1) + \text{Cl}(n-2,k), \text{ and eventually:} \\ &= \text{Cl}(n-1,k-1) + \text{Cl}(n-2,k-1) + \cdots + \text{Cl}(k,k-1) + \text{Cl}(k,k) \\ &= \text{Cl}(n-1,k-1) + \text{Cl}(n-2,k-1) + \cdots + \text{Cl}(k-1,k-1), \end{aligned}$$

because $\mathrm{Cl}(k,k) = \mathrm{Cl}(k-1,k-1) = 1$. Then

$$\begin{aligned}
\mathrm{Cl}(n,2) &= \mathrm{Cl}(n-1,1) + \mathrm{Cl}(n-2,1) + \cdots + \mathrm{Cl}(1,1) \\
&= (6\mathrm{Pa}(n-1,2)+1) + (6\mathrm{Pa}(n-2,2)+1) + \cdots + (6\mathrm{Pa}(2,2)+1) + 1 \\
&= 6(\mathrm{Pa}(n-1,2) + \mathrm{Pa}(n-2,2) + \cdots + \mathrm{Pa}(2,2)) + (n-1).
\end{aligned}$$

By the identity from 1b, $\mathrm{Pa}(n-1,2)+\mathrm{Pa}(n-2,2)+\cdots+\mathrm{Pa}(2,2) = \mathrm{Pa}(n,3)$. Therefore $\mathrm{Cl}(n,2) = 6\mathrm{Pa}(n,3) + n - 1 = 6\mathrm{Pa}(n,3) + \mathrm{Pa}(n-1,1)$.

The general formula is $\mathrm{Cl}(n,k) = 6\mathrm{Pa}(n,k+1) + \mathrm{Pa}(n-1,k-1)$. This formula follows by induction on $n$. If $n = k$, then $\mathrm{Cl}(n,k) = 1$, and $6\mathrm{Pa}(n,k+1) + \mathrm{Pa}(n-1,k-1) = 6 \cdot 0 + 1 = 1$. Then suppose for some $n \geq k$, $\mathrm{Cl}(n,k) = 6\mathrm{Pa}(n,k+1) + \mathrm{Pa}(n-1,k-1)$. It follows that

$$\begin{aligned}
\mathrm{Cl}(n+1,k) &= \mathrm{Cl}(n,k-1) + \mathrm{Cl}(n,k) \\
&= (6\mathrm{Pa}(n,k) + \mathrm{Pa}(n-1,k-2)) + (6\mathrm{Pa}(n,k+1) + \mathrm{Pa}(n-1,k-1)) \\
&= 6(\mathrm{Pa}(n,k) + \mathrm{Pa}(n,k+1)) + (\mathrm{Pa}(n-1,k-2) + \mathrm{Pa}(n-1,k-1)) \\
&= 6\mathrm{Pa}(n+1,k+1) + \mathrm{Pa}(n,k-1).
\end{aligned}$$

**7a.**

$$\begin{array}{ccccccccccc}
 & \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} & \\
\frac{1}{6} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{60} & & \frac{1}{30} & & \frac{1}{6}
\end{array}$$

**7b.** $\mathrm{Le}(17,1) = \mathrm{Le}(16,0) - \mathrm{Le}(17,0) = \frac{1}{17} - \frac{1}{18} = \frac{1}{306}$.

**7c.** $\mathrm{Le}(17,2) = \mathrm{Le}(16,1) - \mathrm{Le}(17,1) = \mathrm{Le}(15,0) - \mathrm{Le}(16,0) - \mathrm{Le}(17,1) = \frac{1}{2448}$.

**8a.** $\mathrm{Le}(n,1) = \mathrm{Le}(n-1,0) - \mathrm{Le}(n,0) = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$.

**8b.** Because $\mathrm{Le}(n,1) = \frac{1}{n} - \frac{1}{n+1}$,

$$\begin{aligned}
\sum_{i=1}^{2011} \mathrm{Le}(i,1) &= \sum_{i=1}^{2011} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{2010} - \frac{1}{2011}\right) + \left(\frac{1}{2011} - \frac{1}{2012}\right) \\
&= 1 - \frac{1}{2012} \\
&= \frac{2011}{2012}.
\end{aligned}$$

**8c.** $\mathrm{Le}(n,2) = \mathrm{Le}(n-1,1) - \mathrm{Le}(n,1) = \frac{1}{n(n-1)} - \frac{1}{n(n+1)} = \frac{2}{(n-1)(n)(n+1)}$. Note that this result appears in the table as a unit fraction because at least one of the integers $n-1, n, n+1$ is even.

**9a.** Extending the result of 8b gives

$$\sum_{i=1}^{n} \mathrm{Le}(i,1) = \frac{1}{1} - \frac{1}{n},$$

so as $n \to \infty$, $\sum_{i=1}^{n} \mathrm{Le}(i,1) \to 1$. This value appears as $\mathrm{Le}(0,0)$, so $n = k = 0$.

**9b.** $n = k = m - 1$.

**9c.** Because in general $\mathrm{Le}(i,m) = \mathrm{Le}(i-1,m-1) - \mathrm{Le}(i,m-1)$, a partial sum can be rewritten as follows:

$$\begin{aligned}
\sum_{i=m}^{n} \mathrm{Le}(i,m) &= \sum_{i=m}^{n} (\mathrm{Le}(i-1,m-1) - \mathrm{Le}(i,m-1)) \\
&= (\mathrm{Le}(m-1,m-1) - \mathrm{Le}(m,m-1)) + (\mathrm{Le}(m,m-1) - \mathrm{Le}(m+1,m-1)) + \cdots \\
&\quad + (\mathrm{Le}(n-1,m-1) - \mathrm{Le}(n,m-1)) \\
&= \mathrm{Le}(m-1,m-1) - \mathrm{Le}(n,m-1).
\end{aligned}$$

Because the values of $\mathrm{Le}(n, m-1)$ get arbitrarily small as $n$ increases (proof: $\mathrm{Le}(i, j) < \mathrm{Le}(i-1, j-1)$ by construction, so $\mathrm{Le}(n, m-1) < \mathrm{Le}(n-m+1, 0) = \frac{1}{n-m+1}$), the limit of these partial sums is $\mathrm{Le}(m-1, m-1)$. So $n = k = m-1$.

**Note:** This result can be extended even further. In fact, for every value of $k < n$,

$$\mathrm{Le}(n, k) = \sum_{i=n+1}^{\infty} \mathrm{Le}(i, k+1).$$

In other words, each entry in Leibniz's triangle is an infinite sum of the entries in the diagonal directly to its right, beginning with the entry below and to the right of the given one.

**10.** Note that $\frac{1}{3} = \mathrm{Le}(2, 0) = \mathrm{Le}(3, 0) + \mathrm{Le}(3, 1)$. Also $\mathrm{Le}(3, 0) = \mathrm{Le}(4, 0) + \mathrm{Le}(4, 1)$ and $\mathrm{Le}(4, 0) = \mathrm{Le}(5, 0) + \mathrm{Le}(5, 1)$. So

$$\begin{aligned} \frac{1}{3} &= \mathrm{Le}(5, 0) + \mathrm{Le}(5, 1) + \mathrm{Le}(4, 1) + \mathrm{Le}(3, 1) \\ &= \frac{1}{6} + \frac{1}{30} + \frac{1}{20} + \frac{1}{12}. \end{aligned}$$

Therefore $a = 6, b = 12, c = 20, d = 30$ is a solution.

On the other hand, a similar analysis yields

$$\begin{aligned} \frac{1}{3} &= \mathrm{Le}(4, 0) + \mathrm{Le}(5, 1) + \mathrm{Le}(5, 2) + \mathrm{Le}(3, 1), \\ &= \frac{1}{5} + \frac{1}{30} + \frac{1}{60} + \frac{1}{12}, \end{aligned}$$

so $a = 5, b = 12, c = 30, d = 60$ is another such solution.

Finally, notice that $\mathrm{Le}(3, 1) = \frac{1}{12} = \mathrm{Le}(11, 0)$, so that $\frac{1}{12}$ can be rewritten as $\mathrm{Le}(12, 0) + \mathrm{Le}(12, 1) = \frac{1}{13} + \frac{1}{156}$. Then

$$\begin{aligned} \frac{1}{3} &= \mathrm{Le}(4, 0) + \mathrm{Le}(4, 1) + \mathrm{Le}(3, 1) \\ &= \mathrm{Le}(4, 0) + \mathrm{Le}(4, 1) + \mathrm{Le}(12, 0) + \mathrm{Le}(12, 1) \\ &= \frac{1}{5} + \frac{1}{20} + \frac{1}{13} + \frac{1}{156}, \end{aligned}$$

yields $a = 5, b = 13, c = 20, d = 156$.

In general, once a triple $a, b, c$ has been found, rewriting (for example) $\frac{1}{a} = \mathrm{Le}(a-1, 0) = \mathrm{Le}(a, 0) + \mathrm{Le}(a, 1) = \frac{1}{a+1} + \frac{1}{a(a+1)}$ creates an appropriate quadruple, although on occasion these values duplicate another value in the quadruple.

**11.** The formula is $\mathrm{Le}(n, k) = \frac{1}{(n+1)\cdot \mathrm{Pa}(n,k)}$, or equivalently, $\frac{1}{(k+1)\cdot \mathrm{Pa}(n+1,k+1)}$. Because $\mathrm{Pa}(n, 0) = \mathrm{Pa}(n, n) = 1$, when $k = 0$ or $k = n$, the formula is equivalent to the definition of $\mathrm{Le}(n, 0) = \mathrm{Le}(n, n) = \frac{1}{n+1}$.

To prove the formula for $1 \le k \le n-1$, use induction on $k$. The base case $k = 0$ was proved above. If the formula holds for a particular value of $k < n$, then it can be extended to the case $k+1$ using two identities:

$$\begin{aligned} \mathrm{Le}(n+1, k+1) &= \mathrm{Le}(n, k) - \mathrm{Le}(n+1, k) \text{ and} \\ (n+1)\mathrm{Pa}(n, k) &= \frac{(n+1)!}{k!(n-k)!}. \end{aligned}$$

Using the first identity and the inductive hypothesis yields

$$\begin{aligned}\mathrm{Le}(n+1,k+1) &= \mathrm{Le}(n,k) - \mathrm{Le}(n+1,k) \\ &= \frac{1}{(n+1)\cdot \mathrm{Pa}(n,k)} - \frac{1}{(n+2)\cdot \mathrm{Pa}(n+1,k)}.\end{aligned}$$

Applying the second identity to the right side of the equation yields:

$$\begin{aligned}\mathrm{Le}(n+1,k+1) &= \frac{k!\,(n-k)!}{(n+1)!} - \frac{k!(n+1-k)!}{(n+2)!} \\ &= \frac{k!\,(n-k)!}{(n+2)!}\,[(n+2)-(n+1-k)] \\ &= \frac{k!\,(n-k)!}{(n+2)!}\,(k+1) \\ &= \frac{(k+1)!\,((n+1)-(k+1))!}{(n+1)!\,(n+2)} \\ &= \frac{1}{(n+2)\mathrm{Pa}(n+1,k+1)}.\end{aligned}$$

## 9 Relay Problems

**Relay 1-1** Compute the number of integers $n$ for which $2^4 < 8^n < 16^{32}$.

**Relay 1-2** Let $T = TNYWR$. Compute the number of positive integers $b$ such that the number $T$ has exactly two digits when written in base $b$.

**Relay 1-3** Let $T = TNYWR$. Triangle $ABC$ has a right angle at $C$, and $AB = 40$. If $AC - BC = T - 1$, compute $[ABC]$, the area of $\triangle ABC$.

**Relay 2-1** Let $x$ be a positive real number such that $\log_{\sqrt{2}} x = 20$. Compute $\log_2 \sqrt{x}$.

**Relay 2-2** Let $T = TNYWR$. Hannah flips two fair coins, while Otto flips $T$ fair coins. Let $p$ be the probability that the number of heads showing on Hannah's coins is greater than the number of heads showing on Otto's coins. If $p = q/r$, where $q$ and $r$ are relatively prime positive integers, compute $q + r$.

**Relay 2-3** Let $T = TNYWR$. In ARMLovia, the unit of currency is the edwah. Janet's wallet contains bills in denominations of 20 and 80 edwahs. If the bills are worth an average of $2T$ edwahs each, compute the smallest possible value of the bills in Janet's wallet.

## 10 Relay Answers

**Answer 1-1** 41

**Answer 1-2** 35

**Answer 1-3** 111

**Answer 2-1** 5

**Answer 2-2** 17

**Answer 2-3** 1020

## 11 Relay Solutions

**Relay 1-1** Compute the number of integers $n$ for which $2^4 < 8^n < 16^{32}$.

**Solution 1-1** $8^n = 2^{3n}$ and $16^{32} = 2^{128}$. Therefore $4 < 3n < 128$, and $2 \leq n \leq 42$. Thus there are **41** such integers $n$.

**Relay 1-2** Let $T = TNYWR$. Compute the number of positive integers $b$ such that the number $T$ has exactly two digits when written in base $b$.

**Solution 1-2** If $T$ has more than one digit when written in base $b$, then $b \leq T$. If $T$ has fewer than three digits when written in base $b$, then $b^2 > T$, or $b > \sqrt{T}$. So the desired set of bases $b$ is $\left\{b \mid \sqrt{T} < b \leq T\right\}$. When $T = 41$, $\left\lfloor\sqrt{T}\right\rfloor = 6$ and so $6 < b \leq 41$. There are $41 - 6 = \mathbf{35}$ such integers.

**Relay 1-3** Let $T = TNYWR$. Triangle $ABC$ has a right angle at $C$, and $AB = 40$. If $AC - BC = T - 1$, compute $[ABC]$, the area of $\triangle ABC$.

**Solution 1-3** Let $AC = b$ and $BC = a$. Then $a^2 + b^2 = 1600$ and $|a - b| = T - 1$. Squaring the second equation yields $a^2 + b^2 - 2ab = (T-1)^2$, so $1600 - 2ab = (T-1)^2$. Hence the area of the triangle is $\frac{1}{2}ab = \frac{1600-(T-1)^2}{4} = 400 - \frac{(T-1)^2}{4}$ or $400 - \left(\frac{T-1}{2}\right)^2$, which for $T = 35$ yields $400 - 289 = \mathbf{111}$.

**Relay 2-1** Let $x$ be a positive real number such that $\log_{\sqrt{2}} x = 20$. Compute $\log_2 \sqrt{x}$.

**Solution 2-1** Use the identity $\log_{b^n} x = \frac{1}{n}\log_b x$ to obtain $\log_2 x = 10$. Then $\log_2 \sqrt{x} = \log_2 x^{1/2} = \frac{1}{2}\log_2 x = \mathbf{5}$.

**Alternate Solution:** Use the definition of log to obtain $x = (\sqrt{2})^{20} = (2^{1/2})^{20} = 2^{10}$. Thus $\log_2 \sqrt{x} = \log_2 2^5 = \mathbf{5}$.

**Alternate Solution:** Use the change of base formula to obtain $\frac{\log x}{\log \sqrt{2}} = 20$, so $\log x = 20 \log \sqrt{2} = 20 \log 2^{1/2} = 10 \log 2$. Thus $x = 2^{10}$, and $\log_2 \sqrt{x} = \log_2 2^5 = \mathbf{5}$.

**Relay 2-2** Let $T = TNYWR$. Hannah flips two fair coins, while Otto flips $T$ fair coins. Let $p$ be the probability that the number of heads showing on Hannah's coins is greater than the number of heads showing on Otto's coins. If $p = q/r$, where $q$ and $r$ are relatively prime positive integers, compute $r - q$.

**Solution 2-2** Because Hannah has only two coins, the only ways she can get more heads than Otto are if she gets 1 (and he gets 0), or she gets 2 (and he gets either 1 or 0).

The probability of Hannah getting exactly one head is $\frac{1}{2}$. The probability of Otto getting no heads is $\frac{1}{2^T}$. So the probability of both events occurring is $\frac{1}{2^{T+1}}$.

The probability of Hannah getting exactly two heads is $\frac{1}{4}$. The probability of Otto getting no heads is still $\frac{1}{2^T}$, but the probability of getting exactly one head is $\frac{T}{2^T}$, because there are $T$ possibilities for which coin is heads. So the probability of Otto getting either 0 heads or 1 head is $\frac{1+T}{2^T}$, and combining that with Hannah's result yields an overall probability of $\frac{1+T}{2^{T+2}}$.

Thus the probability that Hannah flips more heads than Otto is $\frac{1}{2^{T+1}} + \frac{1+T}{2^{T+2}} = \frac{3+T}{2^{T+2}}$. For $T = 5$, the value is $\frac{8}{128} = \frac{1}{16}$, giving an answer of $1 + 16 = \mathbf{17}$.

**Relay 2-3** Let $T = TNYWR$. In ARMLovia, the unit of currency is the edwah. Janet's wallet contains bills in denominations of 20 and 80 edwahs. If the bills are worth an average of $2T$ edwahs each, compute the smallest possible value of the bills in Janet's wallet.

**Solution 2-3** Let $x$ be the number of twenty-edwah bills and $y$ be the number of eighty-edwah bills. Then

$$\begin{aligned}\frac{20x+80y}{x+y} &= 2T \\ 20x+80y &= 2Tx+2Ty \\ (80-2T)y &= (2T-20)x.\end{aligned}$$

In the case where $T = 17$ (and hence $2T = 34$), this equation reduces to $46y = 14x$, or $23y = 7x$. Because 23 and 7 are relatively prime, $23 \mid x$ and $7 \mid y$. Therefore the pair that yields the smallest possible value is $(x, y) = (23, 7)$. Then there are $23 + 7 = 30$ bills worth a total of $23 \cdot 20 + 7 \cdot 80 = 460 + 560 = 1020$ edwahs, and $1020/30 = 34$, as required. The answer is **1020**.

**Alternate Solution:** Consider the equation $\frac{20x+80y}{x+y} = 2T$ derived in the first solution. The identity $\frac{20x+80y}{x+y} = 20 + \frac{60y}{x+y}$ yields the following:

$$\begin{aligned}\frac{60y}{x+y} &= 2T-20 \\ \frac{30y}{x+y} &= T-10 \\ x+y &= \frac{30y}{T-10}.\end{aligned}$$

For the smallest value of $x + y$, both $x + y$ and $y$ will be relatively prime. Thus the smallest value of $x + y$ is $\frac{30}{\gcd(T-10,30)}$, which occurs when $y = \frac{T-10}{\gcd(T-10,30)}$. Substituting $T = 17$, the numbers $T - 10 = 7$ and 30 are relatively prime, so $y = 7$ and $x = 23$, for a total of **1020** edwahs.

# 12 Super Relay

**1.** Suppose that neither of the three-digit numbers $M = \underline{4}\,\underline{A}\,\underline{6}$ and $N = \underline{1}\,\underline{B}\,\underline{7}$ is divisible by 9, but the product $M \cdot N$ **is** divisible by 9. Compute the largest possible value of $A + B$.

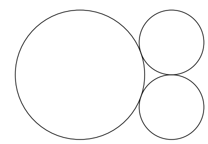
**2.** Let $T = TNYWR$. Each interior angle of a regular $T$-gon has measure $d^\circ$. Compute $d$.

**3.** Let $T = TNYWR$, and let $k$ be the sum of the distinct prime factors of $T$. Suppose that $r$ and $s$ are the two roots of the equation $F_k x^2 + F_{k+1} x + F_{k+2} = 0$, where $F_n$ denotes the $n^{\text{th}}$ Fibonacci number. Compute the value of $(r+1)(s+1)$.

**4.** Let $T = TNYWR$. Compute the product of $-T - i$ and $i - T$, where $i = \sqrt{-1}$.

**5.** Let $T = TNYWR$. Compute the number of positive divisors of the number $20^4 \cdot 11^T$ that are perfect cubes.

**6.** Let $T = TNYWR$. As shown, three circles are mutually externally tangent. The large circle has a radius of $T$, and the smaller two circles each have radius $\frac{T}{2}$. Compute the area of the triangle whose vertices are the centers of the three circles.



**7.** Let $T = TNYWR$, and let $K = (\frac{T}{12})^2$. In the sequence $0.5, 1, -1.5, 2, 2.5, -3, \ldots$, every third term is negative, and the absolute values of the terms form an arithmetic sequence. Compute the sum of the first $K$ terms of this sequence.

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**15.** The sum of the interior angles of an $n$-gon equals the sum of the interior angles of a pentagon plus the sum of the interior angles of an octagon. Compute $n$.

**14.** Let $T = TNYWR$. Compute the value of $x$ that satisfies $\sqrt{20 + \sqrt{T + x}} = 5$.

**13.** Let $T = TNYWR$. Compute $\sin^2 \frac{T\pi}{2} + \sin^2 \frac{(5-T)\pi}{2}$.

**12.** Let $T = TNYWR$. Dennis and Edward each take 48 minutes to mow a lawn, and Shawn takes 24 minutes to mow a lawn. Working together, how many lawns can Dennis, Edward, and Shawn mow in $2 \cdot T$ hours? (For the purposes of this problem, you may assume that after they complete mowing a lawn, they immediately start mowing the next lawn.)

**11.** Let $T = TNYWR$. Susan flips a fair coin $T$ times. Leo has an unfair coin such that the probability of flipping heads is $\frac{1}{3}$. Leo gets to flip his coin the least number of times so that Leo's expected number of heads will exceed Susan's expected number of heads. Compute the number of times Leo gets to flip his coin.

**10.** Let $T = TNYWR$. An isosceles trapezoid has an area of $T + 1$, a height of 2, and the shorter base is 3 units shorter than the longer base. Compute the sum of the length of the shorter base and the length of one of the congruent sides.

**9.** Let $T = TNYWR$. If $\log_2 x^T - \log_4 x = \log_8 x^k$ is an identity for all $x > 0$, compute the value of $k$.

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**8.** Let $A$ be the sum of the digits of the number you will receive from position 7, and let $B$ be the sum of the digits of the number you will receive from position 9. Let $(x, y)$ be a point randomly selected from the interior of the triangle whose consecutive vertices are $(1,1)$, $(B, 7)$ and $(17, 1)$. Compute the probability that $x > A - 1$.

## 13 Super Relay Answers

**Answer 1.** 12

**Answer 2.** 150

**Answer 3.** 2

**Answer 4.** 5

**Answer 5.** 12

**Answer 6.** $72\sqrt{2}$

**Answer 7.** 414

**Answer 8.** $\frac{79}{128}$

**Answer 9.** 27

**Answer 10.** 9.5

**Answer 11.** 16

**Answer 12.** 10

**Answer 13.** 1

**Answer 14.** 14

**Answer 15.** 11

**Answer to the Super-Relay: $\frac{79}{128}$.**

# 14 Super Relay Solutions

**Problem 1.** Suppose that neither of the three-digit numbers $M = \underline{4}\,\underline{A}\,\underline{6}$ and $N = \underline{1}\,\underline{B}\,\underline{7}$ is divisible by 9, but the product $M \cdot N$ **is** divisible by 9. Compute the largest possible value of $A + B$.

**Solution 1.** In order for the conditions of the problem to be satisfied, $M$ and $N$ must both be divisible by 3, but not by 9. Thus the largest possible value of $A$ is 5, and the largest possible value of $B$ is 7, so $A + B = \mathbf{12}$.

**Problem 2.** Let $T = TNYWR$. Each interior angle of a regular $T$-gon has measure $d^\circ$. Compute $d$.

**Solution 2.** From the angle sum formula, $d^\circ = \frac{180^\circ \cdot (T-2)}{T}$. With $T = 12$, $d = \mathbf{150}$.

**Problem 3.** Let $T = TNYWR$, and let $k$ be the sum of the distinct prime factors of $T$. Suppose that $r$ and $s$ are the two roots of the equation $F_k x^2 + F_{k+1} x + F_{k+2} = 0$, where $F_n$ denotes the $n^{\text{th}}$ Fibonacci number. Compute the value of $(r+1)(s+1)$.

**Solution 3.** Distributing, $(r+1)(s+1) = rs + (r+s) + 1 = \frac{F_{k+2}}{F_k} + \left(-\frac{F_{k+1}}{F_k}\right) + 1 = \frac{F_{k+2} - F_{k+1}}{F_k} + 1 = \frac{F_k}{F_k} + 1 = \mathbf{2}$.

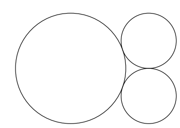
**Problem 4.** Let $T = TNYWR$. Compute the product of $-T - i$ and $i - T$, where $i = \sqrt{-1}$.

**Solution 4.** Multiplying, $(-T - i)(i - T) = -(i + T)(i - T) = -(i^2 - T^2) = 1 + T^2$. With $T = 2$, $1 + T^2 = \mathbf{5}$.

**Problem 5.** Let $T = TNYWR$. Compute the number of positive divisors of the number $20^4 \cdot 11^T$ that are perfect cubes.

**Solution 5.** Let $N = 20^4 \cdot 11^T = 2^8 \cdot 5^4 \cdot 11^T$. If $m \mid N$, then $m = 2^a \cdot 5^b \cdot 11^c$ where $a, b$, and $c$ are nonnegative integers such that $a \leq 8, b \leq 4$, and $c \leq T$. If $m$ is a perfect cube, then $a, b$, and $c$ must be divisible by 3. So $a = 0, 3$, or 6; $b = 0$ or 3, and $c \in \{0, 3, \ldots, 3 \cdot \lfloor T/3 \rfloor\}$. There are a total of $3 \cdot 2 \cdot (\lfloor T/3 \rfloor + 1)$ possible values of $m$. For $T = 5$, $\lfloor T/3 \rfloor + 1 = 2$, so the number of possible values of $m$ is $\mathbf{12}$.

**Problem 6.** Let $T = TNYWR$. As shown, three circles are mutually externally tangent. The large circle has a radius of $T$, and the smaller two circles each have radius $\frac{T}{2}$. Compute the area of the triangle whose vertices are the centers of the three circles.



**Solution 6.** The desired triangle is an isosceles triangle whose base vertices are the centers of the two smaller circles. The congruent sides of the triangle have length $T + \frac{T}{2}$. Thus the altitude to the base has length $\sqrt{\left(\frac{3T}{2}\right)^2 - \left(\frac{T}{2}\right)^2} = T\sqrt{2}$. Thus the area of the triangle is $\frac{1}{2} \cdot \left(\frac{T}{2} + \frac{T}{2}\right) \cdot T\sqrt{2} = \frac{T^2\sqrt{2}}{2}$. With $T = 12$, the area is $\mathbf{72\sqrt{2}}$.

**Problem 7.** Let $T = TNYWR$, and let $K = \left(\frac{T}{12}\right)^2$. In the sequence $0.5, 1, -1.5, 2, 2.5, -3, \ldots$, every third term is negative, and the absolute values of the terms form an arithmetic sequence. Compute the sum of the first $K$ terms of this sequence.

**Solution 7.** The general sequence looks like $x, x+d, -(x+2d), x+3d, x+4d, -(x+5d), \ldots$. The sum of the first three terms is $x - d$; the sum of the second three terms is $x + 2d$; the sum of the third three terms is $x + 5d$, and so on. Thus the sequence of sums of terms $3k-2, 3k-1$, and $3k$ is an arithmetic sequence. Notice that $x = d = 0.5$ and so $x - d = 0$. If there are $n$ triads of terms of the original sequence, then their common difference is 1.5 and their sum is $n \cdot \left(\frac{0 + 0 + (n-1) \cdot 1.5}{2}\right)$. $T = 72\sqrt{2}$, so $K = 72$, and $n = 24$. Thus the desired sum is $\mathbf{414}$.

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**Problem 15.** The sum of the interior angles of an $n$-gon equals the sum of the interior angles of a pentagon plus the sum of the interior angles of an octagon. Compute $n$.

**Solution 15.** Using the angle sum formula, $180^\circ \cdot (n-2) = 180^\circ \cdot 3 + 180^\circ \cdot 6 = 180^\circ \cdot 9$. Thus $n - 2 = 9$, and $n = \mathbf{11}$.

**Problem 14.** Let $T = TNYWR$. Compute the value of $x$ that satisfies $\sqrt{20+\sqrt{T+x}} = 5$.

**Solution 14.** Squaring each side gives $20+\sqrt{T+x} = 25$, thus $\sqrt{T+x} = 5$, and $x = 25-T$. With $T = 11$, $x = \mathbf{14}$.

**Problem 13.** Let $T = TNYWR$. Compute $\sin^2 \frac{T\pi}{2} + \sin^2 \frac{(5-T)\pi}{2}$.

**Solution 13.** Note that $\sin \frac{(5-T)\pi}{2} = \cos(\frac{\pi}{2} - \frac{(5-T)\pi}{2}) = \cos(\frac{T\pi}{2} - 2\pi) = \cos \frac{T\pi}{2}$. Thus the desired quantity is $\sin^2 \frac{T\pi}{2} + \cos^2 \frac{T\pi}{2} = \mathbf{1}$ (independent of $T$).

**Problem 12.** Let $T = TNYWR$. Dennis and Edward each take 48 minutes to mow a lawn, and Shawn takes 24 minutes to mow a lawn. Working together, how many lawns can Dennis, Edward, and Shawn mow in $2 \cdot T$ hours? (For the purposes of this problem, you may assume that after they complete mowing a lawn, they immediately start mowing the next lawn.)

**Solution 12.** Working together, Dennis and Edward take $\frac{48}{2} = 24$ minutes to mow a lawn. When the three of them work together, it takes them $\frac{24}{2} = 12$ minutes to mow a lawn. Thus they can mow 5 lawns per hour. With $T = 1$, they can mow $5 \cdot 2 = \mathbf{10}$ lawns in 2 hours.

**Problem 11.** Let $T = TNYWR$. Susan flips a fair coin $T$ times. Leo has an unfair coin such that the probability of flipping heads is $\frac{1}{3}$. Leo gets to flip his coin the least number of times so that Leo's expected number of heads will exceed Susan's expected number of heads. Compute the number of times Leo gets to flip his coin.

**Solution 11.** The expected number of heads for Susan is $\frac{T}{2}$. If Leo flips his coin $N$ times, the expected number of heads for Leo is $\frac{N}{3}$. Thus $\frac{N}{3} > \frac{T}{2}$, so $N > \frac{3T}{2}$. With $T = 10$, the smallest possible value of $N$ is $\mathbf{16}$.

**Problem 10.** Let $T = TNYWR$. An isosceles trapezoid has an area of $T+1$, a height of 2, and the shorter base is 3 units shorter than the longer base. Compute the sum of the length of the shorter base and the length of one of the congruent sides.

**Solution 10.** Let $x$ be the length of the shorter base of the trapezoid. The area of the trapezoid is $\frac{1}{2} \cdot 2 \cdot (x+x+3) = T+1$, so $x = \frac{T}{2} - 1$. Drop perpendiculars from each vertex of the shorter base to the longer base, and note that by symmetry, the feet of these perpendiculars lie $\frac{3}{2} = 1.5$ units away from their nearest vertices of the trapezoid. Hence the congruent sides have length $\sqrt{1.5^2+2^2} = 2.5$. With $T = 16$, $x = 7$, and the desired sum of the lengths is $\mathbf{9.5}$.

**Problem 9.** Let $T = TNYWR$. If $\log_2 x^T - \log_4 x = \log_8 x^k$ is an identity for all $x > 0$, compute the value of $k$.

**Solution 9.** Note that in general, $\log_b c = \log_{b^n} c^n$. Using this identity yields $\log_2 x^T = \log_{2^2} (x^T)^2 = \log_4 x^{2T}$. Thus the left hand side of the given equation simplifies to $\log_4 x^{2T-1}$. Express each side in base 64: $\log_4 x^{2T-1} = \log_{64} x^{6T-3} = \log_{64} x^{2k} = \log_8 x^k$. Thus $k = 3T - \frac{3}{2}$. With $T = 9.5$, $k = \mathbf{27}$.

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**Problem 8.** Let $A$ be the sum of the digits of the number you will receive from position 7, and let $B$ be the sum of the digits of the number you will receive from position 9. Let $(x, y)$ be a point randomly selected from the interior of the triangle whose consecutive vertices are $(1,1), (B,7)$ and $(17,1)$. Compute the probability that $x > A-1$.

**Solution 8.** Let $P = (1,1)$, $Q = (17,1)$, and $R = (B,7)$ be the vertices of the triangle, and let $X = (B,1)$ be the foot of the perpendicular from $R$ to $\overleftrightarrow{PQ}$. Let $M = (A-1,1)$ and let $\ell$ be the vertical line through $M$; then the problem is to determine the fraction of the area of $\triangle PQR$ that lies to the right of $\ell$.

Note that $B \geq 0$ and $A \geq 0$ because they are digit sums of integers. Depending on their values, the line $\ell$ might intersect any two sides of the triangle or none at all. Each case requires a separate computation. There are two cases where the computation is trivial. First, when $\ell$ passes to the left of or through the leftmost vertex of $\triangle PQR$, which occurs when $A-1 \leq \min(B,1)$, the probability is 1. Second, when $\ell$ passes to the right of or through the rightmost vertex of $\triangle PQR$, which occurs when $A-1 \geq \max(B,17)$, the probability is 0.

The remaining cases are as follows.

**Case 1:** The line $\ell$ intersects $\overline{PQ}$ and $\overline{PR}$ when $1 \le A-1 \le 17$ and $A-1 \le B$.

**Case 2:** The line $\ell$ intersects $\overline{PQ}$ and $\overline{QR}$ when $1 \le A-1 \le 17$ and $A-1 \ge B$.

**Case 3:** The line $\ell$ intersects $\overline{PR}$ and $\overline{QR}$ when $17 \le A-1 \le B$.

Now proceed case by case.

**Case 1:** Let $T$ be the point of intersection of $\ell$ and $\overline{PR}$. Then the desired probability is $[MQRT]/[PQR] = 1-[PMT]/[PQR]$. Since $\triangle PMT \sim \triangle PXR$ and the areas of similar triangles are proportional to the squares of corresponding sides, $[PMT]/[PXR] = (PM/PX)^2$. Since $\triangle PXR$ and $\triangle PQR$ both have height $XR$, their areas are proportional to their bases: $[PXR]/[PQR] = PX/PQ$. Taking the product, $[PMT]/[PQR] = (PM/PX)^2(PX/PQ) = \frac{PM^2}{PX \cdot PQ} = \frac{(A-2)^2}{(B-1)(17-1)}$, and the final answer is

$$\frac{[MQRT]}{[PQR]} = 1 - \frac{[PMT]}{[PQR]} = 1 - \frac{(A-2)^2}{16(B-1)}.$$

**Case 2:** Let $U$ be the point of intersection of $\ell$ and $\overline{QR}$. A similar analysis to the one in the previous case yields

$$\frac{[MQU]}{[PQR]} = \frac{[MQU]}{[XQR]} \cdot \frac{[XQR]}{[PQR]} = \left(\frac{MQ}{XQ}\right)^2 \frac{XQ}{PQ} = \frac{(18-A)^2}{16(17-B)}.$$

**Case 3:** Let $T$ be the point of intersection of $\ell$ and $\overline{PR}$ and let $U$ be the point of intersection of $\ell$ and $\overline{QR}$ as in the previous cases. Let $S$ be the point on $\overline{PR}$ such that $\overline{QS} \perp \overline{PQ}$. Then $\triangle TUR \sim \triangle SQR$, so the areas of these two triangles are proportional to the squares of the corresponding *altitudes* $MX$ and $QX$. Thinking of $\overleftrightarrow{PR}$ as the common base, $\triangle SQR$ and $\triangle PQR$ have a common altitude, so the ratio of their areas is $SR/PR$. Since $\triangle PQS \sim \triangle PXR$, $PS/PR = PQ/PX$ and so $\frac{SR}{PR} = 1 - \frac{PS}{PR} = 1 - \frac{PQ}{PX} = \frac{QX}{PX}$. Therefore the desired probability is

$$\frac{[TUR]}{[PQR]} = \frac{[TUR]}{[SQR]} \cdot \frac{[SQR]}{[PQR]} = \left(\frac{MX}{QX}\right)^2 \frac{QX}{PX} = \frac{(B-A+1)^2}{(B-17)(B-1)}.$$

Using the answers from positions 7 and 9, $A = 4+1+4 = 9$ and $B = 2+7 = 9$. The first case applies, so the probability is

$$1 - \frac{(9-2)^2}{16(9-1)} = 1 - \frac{49}{128} = \mathbf{\frac{79}{128}}.$$

## 15 Tiebreaker Problems

**Problem 1.** Spheres centered at points $P, Q, R$ are externally tangent to each other, and are tangent to plane $\mathcal{M}$ at points $P', Q', R'$, respectively. All three spheres are on the same side of the plane. If $P'Q' = Q'R' = 12$ and $P'R' = 6$, compute the area of $\triangle PQR$.

**Problem 2.** Let $f(x) = x^1 + x^2 + x^4 + x^8 + x^{16} + x^{32} + \cdots$. Compute the coefficient of $x^{10}$ in $f(f(x))$.

**Problem 3.** Compute $\lfloor 100000(1.002)^{10} \rfloor$.

## 16 Tiebreaker Answers

**Answer 1.** $18\sqrt{6}$

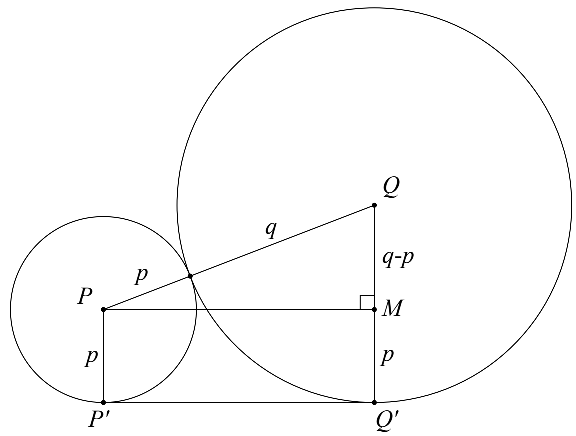
**Answer 2.** 40

**Answer 3.** 102,018

# 17 Tiebreaker Solutions

**Problem 1.** Spheres centered at points $P, Q, R$ are externally tangent to each other, and are tangent to plane $\mathcal{M}$ at points $P', Q', R'$, respectively. All three spheres are on the same side of the plane. If $P'Q' = Q'R' = 12$ and $P'R' = 6$, compute the area of $\triangle PQR$.

**Solution 1.** Let the radii be $p, q, r$ respectively. Looking at a cross-section of the spheres through $\overline{PQ}$ perpendicular to the plane, the points $P', P, Q, Q'$ form a right trapezoid with $\overline{P'P} \perp \overline{P'Q'}$ and $\overline{Q'Q} \perp \overline{P'Q'}$. Draw $\overline{PM}$ perpendicular to $\overline{QQ'}$ as shown.




Then $PP' = MQ' = p$ and $QM = q - p$, while $PQ = p + q$ and $PM = P'Q'$. By the Pythagorean Theorem, $(q-p)^2 + P'Q'^2 = (p+q)^2$, so $q = \frac{(P'Q')^2}{4p}$. Thus $4pq = P'Q'^2 = 12^2$. Similarly, $4pr = P'R'^2 = 6^2$ and $4qr = Q'R'^2 = 12^2$. Dividing the first equation by the third shows that $p = r$ (which can also be inferred from the symmetry of $\triangle P'Q'R'$) and the equation $pr = 9$ yields 3 as their common value; substitute in either of the other two equations to obtain $q = 12$. Therefore the sides of $\triangle PQR$ are $PQ = QR = 12 + 3 = 15$ and $PR = 6$. The altitude to $\overline{PR}$ has length $\sqrt{15^2 - 3^2} = 6\sqrt{6}$, so the triangle's area is $\frac{1}{2}(6)(6\sqrt{6}) = \mathbf{18\sqrt{6}}$.

**Problem 2.** Let $f(x) = x^1 + x^2 + x^4 + x^8 + x^{16} + x^{32} + \cdots$. Compute the coefficient of $x^{10}$ in $f(f(x))$.

**Solution 2.** By the definition of $f$,

$$f(f(x)) = f(x) + (f(x))^2 + (f(x))^4 + (f(x))^8 + \cdots.$$

Consider this series term by term. The first term, $f(x)$, contains no $x^{10}$ terms, so its contribution is 0. The second term, $(f(x))^2$, can produce terms of $x^{10}$ in two ways: as $x^2 \cdot x^8$ or as $x^8 \cdot x^2$. So its contribution is 2.

Now consider the third term:

$$\begin{aligned}(f(x))^4 &= f(x) \cdot f(x) \cdot f(x) \cdot f(x) \\ &= (x^1 + x^2 + x^4 + x^8 + x^{16} + x^{32} + \cdots) \cdot (x^1 + x^2 + x^4 + x^8 + x^{16} + x^{32} + \cdots) \cdot \\ &\quad (x^1 + x^2 + x^4 + x^8 + x^{16} + x^{32} + \cdots) \cdot (x^1 + x^2 + x^4 + x^8 + x^{16} + x^{32} + \cdots).\end{aligned}$$

Each $x^{10}$ term in the product is the result of multiplying four terms whose exponents sum to 10, one from each factor of $f(x)$. Thus this product contains a term of $x^{10}$ for each quadruple of nonnegative integers $(i, j, k, l)$ such that $2^i + 2^j + 2^k + 2^l = 10$; the order of the quadruple is relevant because rearrangements of the integers correspond to choosing terms from different factors. Note that none of the exponents can exceed 2 because $2^3 + 2^0 + 2^0 + 2^0 > 10$. Therefore $i, j, k, l \leq 2$. Considering cases from largest values to smallest yields two basic cases. First, $10 = 4 + 4 + 1 + 1 = 2^2 + 2^2 + 2^0 + 2^0$, which yields $\frac{4!}{2! \cdot 2!} = 6$ ordered quadruples. Second, $10 = 4 + 2 + 2 + 2 = 2^2 + 2^1 + 2^1 + 2^1$, which yields 4 ordered quadruples. Thus the contribution of the $(f(x))^4$ term is $6 + 4 = 10$.

The last term to consider is $f(x)^8$, because $(f(x))^n$ contains no terms of degree less than $n$. An analogous analysis to the case of $(f(x))^4$ suggests that the expansion of $(f(x))^8$ has an $x^{10}$ term for every ordered partition of 10 into a sum of eight powers of two. Up to order, there is only one such partition: $2^1 + 2^1 + 2^0 + 2^0 + 2^0 + 2^0 + 2^0 + 2^0$, which yields $\frac{8!}{6! \cdot 2!} = 28$ ordered quadruples.

Therefore the coefficient of $x^{10}$ is $2 + 10 + 28 = \mathbf{40}$.

**Problem 3.** Compute $\lfloor 100000(1.002)^{10} \rfloor$.

**Solution 3.** Consider the expansion of $(1.002)^{10}$ as $(1+0.002)^{10}$. Using the Binomial Theorem yields the following:

$$(1 + 0.002)^{10} = 1 + \binom{10}{1}(0.002) + \binom{10}{2}(0.002)^2 + \binom{10}{3}(0.002)^3 + \cdots + (0.002)^{10}.$$

However, when $k > 3$, the terms $\binom{10}{k}(0.002)^k$ do not affect the final answer, because $0.002^4 = 0.000000000016 = \frac{16}{10^{12}}$, and the maximum binomial coefficient is $\binom{10}{5} = 252$, so

$$\binom{10}{4}(0.002)^4 + \binom{10}{5}(0.002)^5 + \cdots + (0.002)^{10} < \frac{252 \cdot 16}{10^{12}} + \frac{252 \cdot 16}{10^{12}} + \cdots + \frac{252 \cdot 16}{10^{12}},$$

where the right side of the inequality contains seven terms, giving an upper bound of $\frac{7 \cdot 252 \cdot 16}{10^{12}}$. The numerator is approximately 28000, but $\frac{28000}{10^{12}} = 2.8 \times 10^{-8}$. So even when multiplied by $100000 = 10^5$, these terms contribute at most $3 \times 10^{-3}$ to the value of the expression before rounding.

The result of adding the first four terms ($k = 0$ through $k = 3$) and multiplying by 100,000 is given by the following sum:

$$100000 + 10(200) + 45(0.4) + 120(0.0008) = 100000 + 2000 + 18 + 0.096 = 102018.096.$$

Then the desired quantity is $\lfloor 102018.096 \rfloor = \mathbf{102{,}018}$.