ARML Competition 2023

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Sponsored By:

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1 Team Round

Problem 1. A six-digit natural number is "*sort-of-decreasing*" if its first three digits are in strictly decreasing order and its last three digits are in strictly decreasing order. For example, 821950 and 631631 are sort-of-decreasing but 853791 and 911411 are not. Compute the number of sort-of-decreasing six-digit natural numbers.

Problem 2. For each positive integer N, let $P(N)$ denote the product of the digits of N. For example, $P(8) = 8$, $P(451) = 20$, and $P(2023) = 0$. Compute the least positive integer n such that $P(n + 23) = P(n) + 23$.

Problem 3. Compute the least integer value of the function

$$f(x) = \frac{x^4 - 6x^3 + 2x^2 - 6x + 2}{x^2 + 1}$$

whose domain is the set of all real numbers.

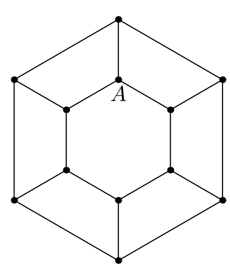
Problem 4. Suppose that noncongruent triangles ABC and XYZ are given such that $AB = XY = 10$, $BC = YZ = 9$, and $\mathrm{m}\angle CAB = \mathrm{m}\angle ZXY = 30°$. Compute $[ABC] + [XYZ]$.

Problem 5. The mean, median, and unique mode of a list of positive integers are three consecutive integers in some order. Compute the least possible sum of the integers in the original list.

Problem 6. David builds a circular table; he then carves one or more positive integers into the table at points equally spaced around its circumference. He considers two tables to be the same if one can be rotated so that it has the same numbers in the same positions as the other. For example, a table with the numbers 8, 4, 5 (in clockwise order) is considered the same as a table with the numbers 4, 5, 8 (in clockwise order), but both tables are different from a table with the numbers 8, 5, 4 (in clockwise order). Given that the numbers he carves sum to 17, compute the number of different tables he can make.

Problem 7. In quadrilateral $ABCD$, $\mathrm{m}\angle B + \mathrm{m}\angle D = 270°$. The circumcircle of $\triangle ABD$ intersects $\overline{CD}$ at point E, distinct from D. Given that $BC = 4$, $CE = 5$, and $DE = 7$, compute the diameter of the circumcircle of $\triangle ABD$.

Problem 8. Suppose that Xena traces a path along the segments in the figure shown, starting and ending at point A. The path passes through each of the eleven vertices besides A exactly once, and only visits A at the beginning and end of the path. Compute the number of possible paths Xena could trace.



Problem 9. Let $i = \sqrt{-1}$. The complex number $z = -142 + 333\sqrt{5}i$ can be expressed as a product of two complex numbers in multiple different ways, two of which are $(57 - 8\sqrt{5}i)(-6 + 5\sqrt{5}i)$ and $(24 + \sqrt{5}i)(-3 + 14\sqrt{5}i)$. Given that $z = -142 + 333\sqrt{5}i$ can be written as $(a + b\sqrt{5}i)(c + d\sqrt{5}i)$, where a, b, c, and d are *positive* integers, compute the lesser of $a + b$ and $c + d$.

Problem 10. Parallelogram $ABCD$ is rotated about A in the plane, resulting in $AB'C'D'$, with D on $\overline{AB'}$. Suppose that $[B'CD] = [ABD'] = [BCC']$. Compute $\tan \angle ABD$.

2 Team Round Answers

Answer 1. 14400

Answer 2. 34

Answer 3. -7

Answer 4. $25\sqrt{3}$

Answer 5. 12

Answer 6. 7711

Answer 7. $\sqrt{130}$

Answer 8. 16

Answer 9. 17

Answer 10. $\sqrt{2}-1$

3 Team Round Solutions

Problem 1. A six-digit natural number is "*sort-of-decreasing*" if its first three digits are in strictly decreasing order and its last three digits are in strictly decreasing order. For example, 821950 and 631631 are sort-of-decreasing but 853791 and 911411 are not. Compute the number of sort-of-decreasing six-digit natural numbers.

Solution 1. If three distinct digits are chosen from the set of digits $\{0, 1, 2, \ldots, 9\}$, then there is exactly one way to arrange them in decreasing order. There are $\binom{10}{3} = 120$ ways to choose the first three digits and 120 ways to choose the last three digits. Thus the answer is $120 \cdot 120 = \mathbf{14400}$.

Problem 2. For each positive integer N, let $P(N)$ denote the product of the digits of N. For example, $P(8) = 8$, $P(451) = 20$, and $P(2023) = 0$. Compute the least positive integer n such that $P(n + 23) = P(n) + 23$.

Solution 2. One can verify that no single-digit positive integer n satisfies the conditions of the problem.

If n has two digits, then $n + 23$ cannot be a three-digit number; this can be verified by checking the numbers $n \geq 88$, because if $n < 88$, then one of the digits of $n + 23$ is 0. Therefore both n and $n + 23$ must be two-digit numbers, so the only possible carry for $n + 23$ will occur in the tens place. If there is a carry for $n + 23$, then $n = \underline{a}\,\underline{8}$ or $n = \underline{a}\,\underline{9}$, while $n + 23 = \underline{(a + 3)}\,\underline{1}$ or $n + 23 = \underline{(a + 3)}\,\underline{2}$, respectively (the case $n = \underline{a}\,\underline{7}$ is omitted because then $P(n + 23) = 0$). In either case, $P(n + 23) < P(n)$ because $a \geq 1$. Otherwise, assume $n = \underline{a}\,\underline{b}$ and $n + 23 = \underline{(a + 2)}\,\underline{(b + 3)}$ is a solution to the given equation, which implies

$$23 = P(n + 23) - P(n) = (a + 2)(b + 3) - ab = 3a + 2b + 6.$$

This means $3a + 2b = 17$, which has solutions $(a, b) = (5, 1)$, $(3, 4)$ as a, b are digits and $b < 7$. The two-digit solutions are $n = 34$ or $n = 51$; thus the least n such that $P(n + 23) = P(n) + 23$ is $n = \mathbf{34}$.

Note: The following shows how to determine all solutions to the given equation. The reader may find this an interesting extension of the conditions of the problem.

Suppose that $n = \underline{a_1}\,\underline{a_2}\,\ldots\,\underline{a_k}$, which implies $P(n) = a_1 a_2 \cdots a_k$. If n and $n + 23$ differ in any digit preceding the hundreds place (where leading digits of 0 are allowed if necessary to compare these digits), then one of the digits of $n+23$ must be 0, which means $P(n+23) = 0$, and thus n would not be a solution to the given equation.

Otherwise, suppose $k > 3$, and let $n + 23 = \underline{a_1}\,\underline{a_2}\,\ldots\,\underline{a_{k-3}}\,\underline{b_{k-2}}\,\underline{b_{k-1}}\,\underline{b_k}$, where b_{k-2}, b_{k-1}, b_k may differ from a_{k-2}, a_{k-1}, a_k, respectively. The given equation implies

$$23 = P(n + 23) - P(n) = (b_{k-2} b_{k-1} b_k - a_{k-2} a_{k-1} a_k) \prod_{j=1}^{k-3} a_j.$$

Because the product is a divisor of 23 and 23 is prime, it follows that the product is equal to 1 and therefore, $a_j = 1$ for all $1 \leq j < k - 2$. This means that the number $n = \underline{a_{k-2}}\,\underline{a_{k-1}}\,\underline{a_k}$ with $n + 23 = \underline{b_{k-2}}\,\underline{b_{k-1}}\,\underline{b_k}$ would be a strictly smaller solution to the given equation. Therefore it suffices to find all solutions for n with at most three digits.

All solutions to $P(n + 23) = P(n) + 23$ will be of the form $n = 34$, $n = 51$, $n = 11\ldots134$, or $n = 11\ldots151$. The least such solution is $n = 34$.

Problem 3. Compute the least integer value of the function

$$f(x) = \frac{x^4 - 6x^3 + 2x^2 - 6x + 2}{x^2 + 1}$$

whose domain is the set of all real numbers.

Solution 3. Use polynomial long division to rewrite $f(x)$ as

$$f(x) = x^2 - 6x + 1 + \frac{1}{x^2+1}.$$

The quadratic function $x^2 - 6x + 1 = (x-3)^2 - 8$ has a minimum of -8, achieved at $x = 3$. The "remainder term" $\frac{1}{x^2+1}$ is always positive. Thus $f(x) > -8$ for all x, so any integer value of $f(x)$ must be at least -7.

When $x = 3$, the remainder term is less than 1, so $f(3)$ is less than -7. But $f(4) = -\frac{34}{5} > -7$, so there must be some value of x between 3 and 4 for which $f(x) = -7$, so the least integer value of $f(x)$ is $\mathbf{-7}$. The reader may note that $f(x) = -7$ when $x \approx 2.097$ and $x \approx 3.970$.

Problem 4. Suppose that noncongruent triangles ABC and XYZ are given such that $AB = XY = 10$, $BC = YZ = 9$, and $m\angle CAB = m\angle ZXY = 30°$. Compute $[ABC] + [XYZ]$.

Solution 4. Because triangles ABC and XYZ are noncongruent yet have two adjacent sides and an angle in common, the two triangles are the two possibilities in the ambiguous case of the Law of Sines. Without loss of generality, let triangle ABC have obtuse angle C and triangle XYZ have acute angle Z so that $m\angle C + m\angle Z = 180°$. Place triangle ABC so that B and Y coincide, and C and Z coincide. Because $m\angle C$ and $m\angle Z$ add up to $180°$, it follows that points X, Z, and A all lie on the same line. The two triangles together then form $\triangle ABX$, where $m\angle BAX = m\angle BXA = 30°$ and $BX = AB = 10$. Therefore the sum of the areas of the two triangles is equal to the area of triangle ABX, which is $\frac{1}{2} \cdot 10 \cdot 10 \cdot \sin(120°) = \frac{5 \cdot 10 \cdot \sqrt{3}}{2} = \mathbf{25\sqrt{3}}$.

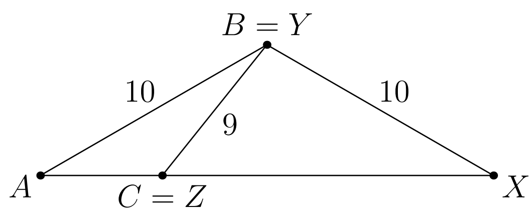


Figure not drawn to scale.

Alternate Solution: As explained above, let $\triangle ABC$ have obtuse angle C and $\triangle XYZ$ have acute angle Z. By the Law of Sines, $\sin(\angle C) = \sin(\angle Z) = \frac{5}{9}$. This implies $m\angle XYZ = \frac{5\pi}{6} - \arcsin\left(\frac{5}{9}\right)$ and $m\angle ABC = \arcsin(\frac{5}{9}) - \frac{\pi}{6}$. The areas of the triangles are $[XYZ] = \frac{1}{2} \cdot 10 \cdot 9 \cdot \sin\left(\frac{5\pi}{6} - \arcsin(\frac{5}{9})\right)$ and $[ABC] = \frac{1}{2} \cdot 10 \cdot 9 \cdot \sin\left(\arcsin(\frac{5}{9}) - \frac{\pi}{6}\right)$. By the angle subtraction rule, it follows that

$$\sin\left(\frac{5\pi}{6} - \arcsin\left(\frac{5}{9}\right)\right) = \sin\left(\frac{5\pi}{6}\right)\cos\left(\arcsin\left(\frac{5}{9}\right)\right) - \cos\left(\frac{5\pi}{6}\right)\sin\left(\arcsin\left(\frac{5}{9}\right)\right) \text{ and}$$
$$\sin\left(\arcsin\left(\frac{5}{9}\right) - \frac{\pi}{6}\right) = \sin\left(\arcsin\left(\frac{5}{9}\right)\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\arcsin\left(\frac{5}{9}\right)\right)\sin\left(\frac{\pi}{6}\right).$$

The sum of the two sines is $\sin\left(\arcsin(\frac{5}{9})\right)\left(\cos(\frac{\pi}{6}) - \cos(\frac{5\pi}{6})\right) = \frac{5}{9} \cdot \sqrt{3}$ because $\sin(\frac{\pi}{6}) = \sin(\frac{5\pi}{6})$. Finally, the sum of the areas of the two triangles is $\frac{1}{2} \cdot 10 \cdot 9 \cdot \frac{5}{9} \cdot \sqrt{3} = 25\sqrt{3}$.

Problem 5. The mean, median, and unique mode of a list of positive integers are three consecutive integers in some order. Compute the least possible sum of the integers in the original list.

Solution 5. One possible list is 1, 1, 3, 7, which has mode 1, median 2, and mean 3. The sum is $1+1+3+7 = 12$. A list with fewer than four numbers cannot produce a median and unique mode that are distinct from each other. To see this, first note that a list with one number has the same median and mode. In a list with two numbers, the mode is not unique if the numbers are different, and if the numbers are the same, the median and mode are equal. In a list of three numbers with a unique mode, the mode must occur twice. Hence the mode is equal to the middle number of the three, which is the median. Thus a list with a median and unique mode that are different from each other must contain at least four numbers.

Now suppose that a list satisfying the given conditions sums to less than 12. The mean must be greater than 1, and because the list contains at least four numbers, the mean must be exactly 2. The median must also be greater than 1, and if the mode is 4, then the sum must be greater than 12. Thus it remains to determine if a mean of 2 with mode 1 and median 3 can be achieved with a list of four or five positive integers. However, having two 1s in the list and a median of 3 forces the remaining numbers in each case to have a sum too large for a mean of 2. The least possible sum is therefore **12**.

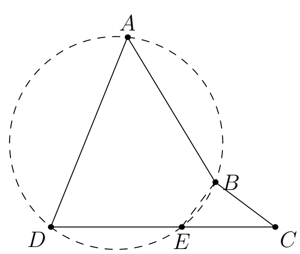
Problem 6. David builds a circular table; he then carves one or more positive integers into the table at points equally spaced around its circumference. He considers two tables to be the same if one can be rotated so that it has the same numbers in the same positions as the other. For example, a table with the numbers 8, 4, 5 (in clockwise order) is considered the same as a table with the numbers 4, 5, 8 (in clockwise order), but both tables are different from a table with the numbers 8, 5, 4 (in clockwise order). Given that the numbers he carves sum to 17, compute the number of different tables he can make.

Solution 6. The problem calls for the number of ordered partitions of 17, where two partitions are considered the same if they are cyclic permutations of each other. Because 17 is prime, each ordered partition of 17 into n parts will be a cyclic permutation of exactly n such partitions (including itself), unless $n = 17$. (If $n = 17$, then all the numbers are 1s, and there is exactly one table David can make.) By the sticks and stones method, the number of ordered partitions of 17 into n nonzero parts is $\binom{16}{n-1}$, and this overcounts the number of tables by a factor of n, except when $n = 17$. Thus the number of possible tables is

$$1 + \sum_{n=1}^{16} \binom{16}{n-1} \cdot \frac{1}{n} = 1 + \sum_{n=1}^{16} \binom{17}{n} \cdot \frac{1}{17} = 1 + \frac{2^{17} - 2}{17} = \mathbf{7711}.$$

Problem 7. In quadrilateral $ABCD$, $\mathrm{m}\angle B + \mathrm{m}\angle D = 270°$. The circumcircle of $\triangle ABD$ intersects $\overline{CD}$ at point E, distinct from D. Given that $BC = 4$, $CE = 5$, and $DE = 7$, compute the diameter of the circumcircle of $\triangle ABD$.

Solution 7. Note that $\mathrm{m}\angle A + \mathrm{m}\angle C = 90°$ in quadrilateral $ABCD$. Because quadrilateral $ABED$ is cyclic, it follows that $\mathrm{m}\angle ADE + \mathrm{m}\angle ABE = 180°$. Moreover, because $\mathrm{m}\angle ABE + \mathrm{m}\angle EBC + \mathrm{m}\angle ADE = 270°$, it follows that $\angle EBC$ is a right angle. Thus $BE = \sqrt{CE^2 - BC^2} = \sqrt{5^2 - 4^2} = 3$. Let $\mathrm{m}\angle BEC = \theta$; then $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$.



Applying the Law of Cosines to $\triangle BED$ yields

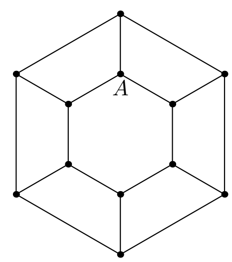
$$BD^2 = 3^2 + 7^2 - 2 \cdot 3 \cdot 7\cos(180° - \theta) = 3^2 + 7^2 + 2 \cdot 3 \cdot 7\cos\theta = \frac{416}{5}.$$

Thus $BD = \frac{4\sqrt{26}}{\sqrt{5}}$. Let R be the circumradius of $\triangle ABD$ and $\triangle BED$. Then the requested diameter is $2R$, and applying the Law of Sines to $\triangle BED$ yields

$$2R = \frac{BD}{\sin(180^\circ - \theta)} = \frac{BD}{\sin\theta} = \frac{4\sqrt{26}}{\sqrt{5}} \cdot \frac{5}{4} = \mathbf{\sqrt{130}}.$$

Alternate Solution: Proceed as above to conclude that $\angle EBC$ is a right angle and $BE = 3$. Extend $\overline{BC}$ past B to intersect the circle a second time at F. By Power of a Point from C, it follows that $CE \cdot CD = CB \cdot CF$, from which $CF = \frac{5 \cdot 12}{4} = 15$, and $BF = 11$. Because $\angle EBF$ is also a right angle, it follows that $\overline{EF}$ is a diameter of the circle and that $EF = \sqrt{3^2 + 11^2} = \sqrt{130}$.

Problem 8. Suppose that Xena traces a path along the segments in the figure shown, starting and ending at point A. The path passes through each of the eleven vertices besides A exactly once, and only visits A at the beginning and end of the path. Compute the number of possible paths Xena could trace.



Solution 8. Count the number of complete paths that pass through all vertices exactly once (such a path is called a *Hamiltonian path*). The set of vertices can be split into two rings:

$$\mathcal{I} = \{A_1, A_2, \ldots, A_6\} \text{ (i.e., the inner ring)}, \quad \mathcal{O} = \{B_1, B_2, \ldots, B_6\} \text{ (i.e., the outer ring)}.$$

where $A_1 = A$. The two rings are connected by the edges $E = \{A_1B_1, A_2B_2, \ldots, A_6B_6\}$. Each vertex in the figure has exactly three edges joining it with the neighboring vertices. Also note that any closed loop must use exactly two edges (out of three) for each vertex.

Further note that a loop must use at least one edge from E to move from one ring to the other. Consider two cases: the loop uses all six edges from E, or it uses some but not all of them.

If all edges from E are used, there are two possible undirected loops. It is not possible to use both edges A_1A_2 and B_1B_2, so either A_1A_2 or B_1B_2 will be used. This choice determines how the entire loop is constructed.

If not all edges from E are used, then there must be some i for which the loop uses the edge A_iB_i and does not use $A_{i+1}B_{i+1}$ (where A_jB_j represents $A_{j-6}B_{j-6}$ if $7 \le j \le 12$). Because A_{i+1} is only connected to three other vertices, the loop must use A_iA_{i+1} and $A_{i+1}A_{i+2}$, and similarly must use B_iB_{i+1} and $B_{i+1}B_{i+2}$. This also precludes using $A_{i+2}B_{i+2}$, because doing so would close the loop before it visits all 12 vertices. Therefore the loop must also use $A_{i+2}A_{i+3}$ and $B_{i+2}B_{i+3}$, which now precludes using $A_{i+3}B_{i+3}$. This continues to force the structure of the loop until it closes by using $A_{i+5}B_{i+5} = A_{i-1}B_{i-1}$. Hence the loop must use exactly two edges from E, and they must be consecutive: $A_{i-1}B_{i-1}$ and A_iB_i. There are 6 ways to choose those two consecutive edges, so there are 6 possible undirected loops in this case.

The forced path is a loop, and the only way the given conditions are satisfied if $A_{i+k} = A_{i-1}$ and $B_{i+k} = B_{i-1}$. Hence the loop must use (precisely) two consecutive edges from E: $A_{i-1}B_{i-1}$ and A_iB_i. There are 6 ways to choose two consecutive edges, so there are 6 possible undirected loops in this case.

Each undirected loop can be traced in two ways, and thus the number of ways for Xena to trace the path is $(6 + 2) \cdot 2 = \mathbf{16}$.

Problem 9. Let $i = \sqrt{-1}$. The complex number $z = -142 + 333\sqrt{5}i$ can be expressed as a product of two complex numbers in multiple different ways, two of which are $\left(57 - 8\sqrt{5}i\right)\left(-6 + 5\sqrt{5}i\right)$ and $\left(24 + \sqrt{5}i\right)\left(-3 + 14\sqrt{5}i\right)$. Given that $z = -142 + 333\sqrt{5}i$ can be written as $\left(a + b\sqrt{5}i\right)\left(c + d\sqrt{5}i\right)$, where a, b, c, and d are *positive* integers, compute the lesser of $a + b$ and $c + d$.

Solution 9. Multiply each of the given parenthesized expressions by its complex conjugate to obtain

$$\begin{aligned} 142^2 + 5 \cdot 333^2 &= \left(57^2 + 5 \cdot 8^2\right)\left(6^2 + 5 \cdot 5^2\right) \\ &= \left(24^2 + 5 \cdot 1^2\right)\left(3^2 + 5 \cdot 14^2\right) \\ &= \left(a^2 + 5b^2\right)\left(c^2 + 5d^2\right). \end{aligned}$$

The expression on the second line is equal to $581 \cdot 989 = 7 \cdot 83 \cdot 23 \cdot 43$ (one can perhaps factor 989 a little faster by noting that 23 divides $6^2 + 5 \cdot 5^2 = 7 \cdot 23$ but not 581, so it must divide 989). Thus $a^2 + 5b^2$ and $c^2 + 5d^2$ must be a factor pair of this number. It is not possible to express 1, 7, 23, 43, or 83 in the form $x^2 + 5y^2$ for integers x, y.

Let $N = a^2 + 5b^2$, and without loss of generality, assume that 7 divides N. From the above analysis, N must be $7 \cdot 23$, $7 \cdot 43$, or $7 \cdot 83$. By direct computation of checking all positive integers b less than $\sqrt{\frac{N}{5}}$, the only possibilities for (a, b) are:

- when $N = 7 \cdot 23$, either $(9, 4)$ or $(6, 5)$;
- when $N = 7 \cdot 43$, either $(16, 3)$ or $(11, 6)$; and
- when $N = 7 \cdot 83$, either $(24, 1)$ or $(9, 10)$.

Next, observe that

$$\frac{-142 + 333\sqrt{5}i}{a + b\sqrt{5}i} = \frac{(-142a + 1665b) + (333a + 142b)\sqrt{5}i}{N}$$

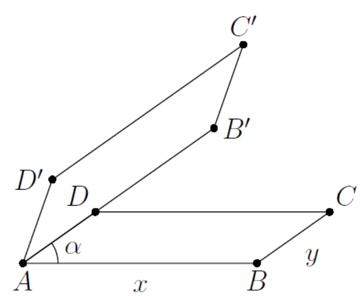
must equal $c + d\sqrt{5}i$, so N must divide $-142a + 1665b$ and $333a + 142b$. But

- 7 does not divide $333 \cdot 9 + 142 \cdot 4$ or $333 \cdot 6 + 142 \cdot 5$;
- 43 does not divide $333 \cdot 16 + 142 \cdot 3$; and
- 83 does not divide $333 \cdot 9 + 142 \cdot 10$.

Thus the only candidates are $(a, b) = (11, 6)$ and $(a, b) = (24, 1)$. Note that $(24, 1)$ yields the second factorization given in the problem statement, which has a negative real part in one of its factors. Thus the only remaining candidate for (a, b) is $(11, 6)$, which yields $(c, d) = (28, 15)$, thus the answer is $11 + 6 = \mathbf{17}$.

Problem 10. Parallelogram $ABCD$ is rotated about A in the plane, resulting in $AB'C'D'$, with D on $\overline{AB'}$. Suppose that $[B'CD] = [ABD'] = [BCC']$. Compute $\tan\angle ABD$.

Solution 10. Let $AB = x$, $BC = y$, and $m\angle A = \alpha$.



It then follows that $[ABD'] = \frac{xy\sin 2\alpha}{2}$ and $[B'CD] = \frac{x(x-y)\sin\alpha}{2}$.

Because $\overline{BC}$, $\overline{AB'}$, and $\overline{D'C'}$ are all parallel, it follows that $\triangle BCC'$ and $\triangle BCD'$ have the same height with respect to base $\overline{BC}$, and thus $[BCC'] = [BCD']$. Therefore $[BCD'] = [ABD']$, and it follows that triangles BCD' and ABD' have the same height with respect to base $\overline{BD'}$. Thus A and C are equidistant from $\overleftrightarrow{BD'}$.

This implies that the midpoint of $\overline{AC}$ lies on $\overleftrightarrow{BD'}$. But $\overleftrightarrow{BD}$ also passes through the midpoint of $\overline{AC}$ by parallelogram properties, so it follows that D must lie on $\overline{BD'}$. This implies that $[ABD']$ must also equal $\frac{y^2\sin\alpha}{2} + \frac{xy\sin\alpha}{2} = \frac{(xy+y^2)\sin\alpha}{2}$.

This implies that $x(x-y)\sin\alpha = xy\sin 2\alpha = (xy+y^2)\sin\alpha$, which implies $x : y = \sqrt{2}+1$ and $\sin\alpha = \cos\alpha = \frac{\sqrt{2}}{2}$. Finally, from right triangle $D'AB$ with legs in the ratio $1 : \sqrt{2}+1$, it follows that $\tan(\angle ABD) = \tan(\angle ABD') = \boldsymbol{\sqrt{2}-1}$.

4 Power Round 2023: A Perfectly Cromulent Power Question

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

In a sequence of n consecutive positive integers, where $n > 1$, an element of the sequence is said to be *cromulent* if it is relatively prime to all other numbers in the sequence. Every element of a sequence with $n = 2$ is cromulent because any two consecutive integers are relatively prime to each other.

1. List a sequence of three consecutive positive integers with

- **a.** exactly one cromulent element; [1 pt]
- **b.** three cromulent elements. [1 pt]

2. List a sequence of four consecutive positive integers with

- **a.** exactly one cromulent element; [1 pt]
- **b.** two cromulent elements. [1 pt]

3. List a sequence of five consecutive positive integers with

- **a.** exactly one cromulent element; [1 pt]
- **b.** exactly two cromulent elements; [1 pt]
- **c.** three cromulent elements. [1 pt]

4. Show that any two consecutive odd positive integers are relatively prime to each other. [2 pts]

5.
- **a.** Show that every sequence of three consecutive positive integers contains a cromulent element. [2 pts]
- **b.** Show that every sequence of four consecutive positive integers contains a cromulent element. [2 pts]
- **c.** Show that every sequence of five consecutive positive integers contains a cromulent element. [3 pts]

6. Find the maximum and minimum possible number of cromulent elements in a sequence of n consecutive positive integers with

- **a.** $n = 6$; [3 pts]
- **b.** $n = 7$. [3 pts]

7. Prove that there is at least one cromulent element in every sequence of n consecutive positive integers with

- **a.** $n = 8$; [4 pts]
- **b.** $n = 9$. [4 pts]

8.
- **a.** Find two sequences of 17 consecutive positive integers with no cromulent elements. [6 pts]
- **b.** Compute the least possible element of such a sequence of length 17. [2 pts]

9. The goal of this problem is to prove the following claim:

Claim: For $k \geq 300$, there is a sequence of k consecutive integers with no cromulent elements.

Let $\pi(n)$ denote the number of primes less than or equal to n. You may use the following lemma without proof in the problems that follow.

Lemma: For $x \geq 75$, $\pi(2x) - \pi(x) \geq 2\lfloor \log_2(x) \rfloor + 2$.

One argument for proving the above claim begins as follows. Let $m = \lfloor \frac{k}{4} \rfloor \geq 75$, let $p_1, p_2, \ldots, p_r$ denote the primes in the set $\{1, 2, \ldots, m\}$ (note that $p_1 = 2$), and let p_{r+1}, p_{r+2}, $\ldots$, p_s denote the primes in the set $\{m+1, m+2, \ldots, 2m\}$.

a. Prove that if each of the k consecutive integers in the sequence is divisible by at least one of the primes $p_1, p_2, \ldots, p_s$, then the sequence has no cromulent elements. [2 pts]

Let x be a solution to the system of congruences $x \equiv 1 \pmod{2}$ and $x \equiv 0 \pmod{p_2 p_3 \cdots p_r}$. Then the integers

$$x - 2m,\ x - 2m + 2,\ \ldots,\ x - 2,\ x,\ x + 2,\ \ldots,\ x + 2m - 2,\ x + 2m$$

form a sequence of $2m + 1$ consecutive odd integers of the form $x \pm 2y$, where y varies from 0 to m.

b. Prove that every number in the above sequence, except those in which y is a power of 2, is divisible by one of the primes $p_2, \ldots, p_r$. [2 pts]

Let $t = \lfloor \log_2(m) \rfloor + 1$. Then the elements of the above list not yet accounted for are $x \pm 2, x \pm 2^2, \ldots, x \pm 2^t$.

c. Prove that there are at least $2t$ primes in the set $\{m+1, m+2, \ldots, 2m\}$. [1 pt]

It therefore follows that $s - r \geq 2t$, hence $p_{r+2t} \leq p_s$.

d. Show that there exists an integer x such that

$$x \equiv 1 \pmod{2},$$
$$x \equiv 0 \pmod{p_2 p_3 \cdots p_r},$$

and for each $u \in \{1, \ldots, t\}$,

$$x + 2^u \equiv 0 \pmod{p_{r+u}}, \text{ and}$$
$$x - 2^u \equiv 0 \pmod{p_{r+t+u}}.$$

[2 pts]

e. Let x be a solution to the system of congruences in part (d). Show that the sequence of $4m+3$ consecutive integers

$$x - 2m - 1,\ x - 2m,\ x - 2m + 1,\ \ldots,\ x - 1,\ x,\ x + 1,\ \ldots,\ x + 2m + 1$$

has no cromulent elements. [3 pts]

f. Show that for $k \geq 300$, there is a sequence of k consecutive integers with no cromulent elements. [2 pts]

5 Power Round Solutions

1. **a.** There are many examples of sequences of three consecutive positive integers with exactly one cromulent element. One such sequence is 2, 3, 4. Note that 2 and 4 share the common factor of 2, while 3 is prime. More generally, such a sequence will be of the form $2n$, $2n+1$, $2n+2$ for some positive integer n.

b. There are many examples of sequences of three consecutive positive integers with three cromulent elements. One such sequence is 1, 2, 3. More generally, such a sequence will be of the form $2n-1$, $2n$, $2n+1$ for some positive integer n.

2. **a.** There are many examples of sequences of four consecutive positive integers with exactly one cromulent element. One such sequence is 3, 4, 5, 6. Note that 4 and 6 share the common factor of 2, and that 3 and 6 share the common factor of 3, while 5 is prime and thus cromulent.

b. There are many examples of sequences of four consecutive positive integers with two cromulent elements. One such sequence is 2, 3, 4, 5. Note that 2 and 4 share the common factor of 2, while 3 and 5 are prime and thus cromulent.

3. **a.** There are many examples of sequences of five consecutive positive integers with exactly one cromulent element. One such sequence is 2, 3, 4, 5, 6. Note that 2, 4, and 6 share the common factor of 2, and that 3 and 6 share the common factor of 3, while 5 is prime and thus cromulent.

b. There are many examples of sequences of five consecutive positive integers with two cromulent elements. One such sequence is 4, 5, 6, 7, 8. Note that 4, 6, and 8 share the common factor of 2, while 5 and 7 are prime and thus cromulent.

c. There are many examples of sequences of five consecutive positive integers with three cromulent elements. One such sequence is 1, 2, 3, 4, 5. Note that 2 and 4 share the common factor of 2, while 3 and 5 prime, thus 1, 3, and 5 are cromulent.

4. Two consecutive odd integers differ by 2. Thus any common divisor of the two integers must also divide 2. However, the only prime divisor of 2 is 2, and neither of the consecutive odd integers is a multiple of 2. Therefore the consecutive odd integers are relatively prime to each other.

5. **a.** Consider the sequence $a-1$, a, $a+1$. Note that a is relatively prime to $a-1$ and $a+1$ because any common divisor must divide their difference, and the differences are both 1. Thus a is a cromulent element of the sequence.

b. Consider a sequence of four consecutive integers. Notice that two of these integers will be odd, and by problem 4, these are relatively prime to each other. At most one of these two odd integers can be a multiple of 3. Let a be an odd integer in the sequence that is not a multiple of 3. Then a is cromulent by the following reasoning. Note that the difference between a and any other element of the sequence is at most 3. Thus if a shared a common factor greater than 1 with some other element of the sequence, this factor would have to be 2 or 3. But because a is odd and not a multiple of 3, it follows that a is cromulent.

c. Consider a sequence of five consecutive integers. Exactly one number in such a sequence will be a multiple of 5, but that number could also be a multiple of 2 and hence share a common factor with at least one other number in the sequence. There are several cases to consider, namely whether the sequence starts with an even number or an odd number.

If the sequence starts with an even number, then the second and fourth numbers are both odd, and at least one of them is not a multiple of 3 and hence is relatively prime to all other numbers in the sequence because it is neither a multiple of 2 nor 3 and hence is at least 5 away from the nearest integer with a common factor.

If the sequence starts with an odd number, then it again contains an odd number that is not a multiple of 3 and hence is relatively prime to all other numbers in the sequence. In fact, it contains two such numbers if the first or last number is a multiple of 3, and if the middle number is a multiple of 3 then all three odd elements are cromulent.

6. **a.** The minimum number is 1 and the maximum number is 2. One example of a sequence of length 6 with 1 cromulent element is 5, 6, 7, 8, 9, 10, where 7 is the cromulent element. To show that it is not possible for a sequence of 6 consecutive elements to have 0 cromulent elements, consider two cases. If the sequence begins with an even number, that number is not cromulent, and one of the other five elements must be cromulent by the argument in the solution to 5(c). A similar argument establishes that one element must be cromulent if the sequence of 6 begins with an odd number (and thus ends in an even number).

One example of a sequence of length 6 with 2 cromulent elements is 1, 2, 3, 4, 5, 6, where 1 and 5 are both cromulent.

To prove that a sequence of length 6 cannot have three cromulent elements, consider that the cromulent elements would all have to be odd, and one of those three would be a multiple of 3. Because one of the even elements must also be a multiple of 3, it is not possible for all three odd elements to be cromulent.

b. The minimum number is 1 and the maximum number is 3. One example of a sequence of length 7 with 1 cromulent element is 4, 5, 6, 7, 8, 9, 10, where 7 is the cromulent element. To show that it is not possible for such a sequence to have zero cromulent elements, consider two cases. If the sequence begins with an even number, then it contains 3 odd numbers. At most one of these is divisible by 3, and at most one is divisible by 5, so one of the odd numbers must be divisible by neither 3 nor 5. This odd number differs by at most 6 from each other element of the sequence, so the only prime factors it can share with another element of the sequence are 2, 3, and 5. Because it is divisible by none of these primes, it follows that the odd number in question is cromulent. Similarly, if the sequence begins with an odd number, then it contains 4 odd numbers; at most two of these are divisible by 3, and at most one is divisible by 5, so again, one odd number in the sequence must be divisible by neither 3 nor 5. By the same argument, this element is cromulent.

One example of a sequence of length 7 with 3 cromulent elements is 1, 2, 3, 4, 5, 6, 7, where 1, 5, and 7 are all cromulent.

To prove that a sequence of length 7 cannot have four cromulent elements, consider that the cromulent elements would all have to be odd. At least one of these four odd elements must be a multiple of 3. Because one of the even elements must also be a multiple of 3, it is thus not possible for all four odd elements to be cromulent.

7. a. In any sequence of eight consecutive integers, four of them will be even and hence not cromulent. The primes 3, 5, and 7 are the only numbers that could possibly divide more than one number in the sequence, with 5 and 7 each dividing at most one of the four odd numbers. Note that for 7 to divide two numbers in the sequence, they must be the first and last numbers because only those differ by 7. The prime 3 can divide two odd numbers in the sequence, but only if they differ by 6, which means they must be either the 1st and 7th or the 2nd and 8th numbers in the list. Therefore one of the two odd multiples of 3 would also be the only candidate to be an odd multiple of 7 when there are two multiples of 7, and so at least one of the four odd numbers must be cromulent.

b. In a sequence of nine consecutive integers, again the only primes to check are 2, 3, 5, and 7. If there are five even numbers in the sequence, then at least one of the odd numbers is cromulent by similar reasoning to the solution to 7(a). The difference is that in this case the multiples of 7 can be either the 1st and 8th or the 2nd and 9th numbers, while the odd multiples of 3 must be the 2nd and 8th numbers (because the 1st number is even) and hence overlap with the multiples of 7 again. If instead there are four even numbers in the sequence, then the 1st, 3rd, 5th, 7th, and 9th numbers are odd. Now 7 divides at most one of these odd numbers, 5 divides at most one, and 3 divides at most two, leaving at least one odd cromulent element.

8. a. For a sequence of length 17, the primes to check are 2, 3, 5, 7, 11, and 13. By making the first number a multiple of 2, 3, 7, and 13, and the last number a multiple of 2, 5, and 11, such a sequence can be $2 \cdot 3 \cdot 7 \cdot 13 \cdot a$, $5b$, $2c$, $3d$, $2e$, $11f$, $2 \cdot 3 \cdot 5 \cdot g$, $7h$, $2i$, $3j$, $2k$, 5ℓ, $2 \cdot 3 \cdot m$, $13 \cdot n$, $2 \cdot 7 \cdot p$, $3q$, $2 \cdot 5 \cdot 11 \cdot r$ for positive integers $a, b, c, \ldots, r$, so it contains no cromulent elements. Because the 2nd and 6th elements of the above sequence are multiples of 5 and 11, respectively, it follows that the first element of the sequence must be a multiple of 546 that is also 4 (mod 5) and 6 (mod 11), so 2184 works, as does $2184 + 30030s$ for any positive integer s by the Chinese Remainder Theorem. Similarly, the first number could be a multiple of 2, 5, and 11, and the last number a multiple of 2, 3, 7, and 13, which means such a sequence could start with 27830 or $27830 + 30030t$ for any positive integer t.

b. From the argument in the solution to 8(a), the least possible element of such a sequence of length 17 is 2184.

9. **a.** Let the sequence of k integers be $a_1, a_2, \ldots, a_k$, where $a_n = a_1 + (n-1)$ for $1 \le n \le k$. For any $a_j \in \{a_1, a_2, \ldots, a_{2m}\}$, suppose that a_j is divisible by some $p \in \{p_1, p_2, \ldots, p_s\}$. Then $a_{j+p} = a_j + p$, so a_{j+p} is also divisible by p. Hence neither a_j nor a_{j+p} is cromulent because both numbers are divisible by p (also note that $j + p < 4m \le k$, so a_{j+p} is one of the k integers in the sequence). Thus none of $a_1, a_2, \ldots, a_{2m}$ are cromulent. Similarly, for any $a_{j'} \in \{a_{2m+1}, a_{2m+2}, \ldots, a_k\}$, suppose that $a_{j'}$ is divisible by some $p' \in \{p_1, p_2, \ldots, p_s\}$. Then $a_{j'-p'} = a_{j'} - p'$ is also divisible by p' (also note that $1 < j' - p' \le k - 2$, so $a_{j'-p'}$ is one of the k integers in the sequence). Thus none of $a_{2m+1}, a_{2m+2}, \ldots, a_k$ are cromulent, completing the proof.

b. Note that $x \equiv 1 \pmod 2$ implies that x is odd. Because $x \equiv 0 \pmod{p_2p_3\cdots p_r}$, write $x = qp_2p_3\cdots p_r$ for some odd integer q. Then x is divisible by all of the primes $p_2, \ldots, p_r$. The desired result will first be established for numbers of the form $x + 2y$, with $1 \le y \le m$. Because x is divisible by each of $p_2, \ldots, p_r$, it suffices to prove that y is divisible by one of $p_2, \ldots, p_r$ unless y is a power of 2. Note that because $p_2, \ldots, p_r$ are all odd, if y is a power of 2 (including $2^0 = 1$), then $2y$ cannot be divisible by any of $p_2, \ldots, p_r$. On the other hand, if y has an odd factor greater than 1, by the Fundamental Theorem of Arithmetic, y must be divisible by at least one of $p_2, \ldots, p_r$ (note that $p_2 = 3$ and $p_r \le m$). Finally, because the desired result holds for $x+2y$, with $1 \le y \le m$, it also holds for $x - 2y$ because $x - 2y = (x+2y) - 4y$, thus completing the proof.

c. The definition of the $\pi(\cdot)$ function implies that the number of primes in the set $\{m+1, m+2, \ldots, 2m\}$ is $\pi(2m) - \pi(m)$. Because $m \ge 75$, the lemma can be applied and implies that

$$\pi(2m) - \pi(m) \ge 2\lfloor \log_2(m) \rfloor + 2 = 2t,$$

hence the desired conclusion follows.

d. Note that the moduli of the given congruences:

$$2, \quad p_2p_3\cdots p_r, \quad p_{r+u}, \quad p_{r+t+u} \quad (u \in \{1, \ldots, t\})$$

are pairwise relatively prime. The Chinese Remainder Theorem can therefore be applied to the given system, and implies that the given system of congruences has solutions for x, as desired.

e. Let x be a solution to the system from problem 9(d). Then each element of the subsequence of $2m+2$ elements

$$x - 2m - 1,\ x - 2m + 1,\ \ldots,\ x - 1,\ x + 1,\ \ldots,\ x + 2m + 1 \tag{$*$}$$

is even, hence no element of $(*)$ is cromulent. The remaining elements of the given sequence are

$$x - 2m,\ x - 2m + 2,\ \ldots,\ x - 2,\ x,\ x + 2,\ \ldots,\ x + 2m - 2,\ x + 2m, \tag{$\dagger$}$$

and are all odd. Note that $(\dagger)$ is precisely the sequence considered in problem 9(b). Let $x' = x \pm 2y$ be an element of $(\dagger)$, where $0 \le y \le m$. If y is not a power of 2, then the result of problem 9(b) implies that x' is divisible by one of $p_2, \ldots, p_r$. On the other hand, if $y = 2^w$ for some integer $0 \le w \le t-1$, then according to the third congruence of problem 9(d), $x+2y$ is divisible by p_{r+w+1}, and according to the fourth congruence of problem 9(d), $x-2y$ is divisible by $p_{r+t+w+1}$. Also note that because $r+w+1 < r+t+w+1 \le r+2t \le s$, it follows that p_{r+w+1} and $p_{r+t+w+1}$ are distinct primes among the s primes less than $2m$. Hence the result of problem 9(a) applies, and thus the given sequence has no cromulent elements.

f. Note that $m = \left\lfloor \frac{k}{4} \right\rfloor$ implies that $k \le 4m + 3$. Now consider any sequence $\mathcal{S}$ of k consecutive integers, taken from the sequence of $4m+3$ consecutive integers

$$x - 2m - 1,\ x - 2m,\ x - 2m + 1,\ \ldots,\ x - 1,\ x,\ x + 1,\ \ldots,\ x + 2m + 1.$$

As proven in problem 9(e), each of these $4m+3$ integers is divisible by at least one of the s primes less than or equal to $2m$. When a shortened sequence of k consecutive integers is chosen from among the original $4m+3$ integers, it remains true that each of them is divisible by a prime less than or equal to $2m$. Therefore the result of problem 9(a) again implies that the shortened sequence has no cromulent element, as desired.

6 Individual Round

Problem 1. Let $N = 888{,}888 \times 9{,}999{,}999$. Compute the sum of the digits of N.

Problem 2. Five equilateral triangles are drawn in the plane so that no two sides of any of the triangles are parallel. Compute the maximum number of points of intersection among all five triangles.

Problem 3. Let S be the set of four-digit positive integers for which the sum of the squares of their digits is 17. For example, $2023 \in S$ because $2^2 + 0^2 + 2^2 + 3^2 = 17$. Compute the median of S.

Problem 4. Let $EUCLID$ be a hexagon inscribed in a circle of radius 5. Given that $EU = UC = LI = ID = 6$, and $CL = DE$, compute CL.

Problem 5. The *ARMLLexicon* consists of 10 letters: $\{A, R, M, L, e, x, i, c, o, n\}$. A *palindrome* is an ordered list of letters that read the same backwards and forwards; for example, *MALAM*, *n*, *oncecno*, and *MoM* are palindromes. Compute the number of 15-letter palindromes that can be spelled using letters in the ARMLLexicon, among which there are four consecutive letters that spell out *ARML*.

Problem 6. Let 10^y be the product of all real numbers x such that $\log x = \dfrac{3 + \lfloor (\log x)^2 \rfloor}{4}$. Compute y.

Problem 7. The solutions to the equation $x^2 - 180x + 8 = 0$ are r_1 and r_2. Compute

$$\frac{r_1}{\sqrt[3]{r_2}} + \frac{r_2}{\sqrt[3]{r_1}}.$$

Problem 8. Circle ω is tangent to parallel lines ℓ_1 and ℓ_2 at A and B respectively. Circle ω_1 is tangent to ℓ_1 at C and to ω externally at P. Circle ω_2 is tangent to ℓ_2 at D and to ω externally at Q. Circles ω_1 and ω_2 are also externally tangent to each other. Given that $AQ = 12$ and $DQ = 8$, compute CD.

Problem 9. Given quadrilateral $ARML$ with $AR = 20$, $RM = 23$, $ML = 25$, and $AM = 32$, compute the number of different integers that could be the perimeter of $ARML$.

Problem 10. Let $\mathcal{S}$ denote the set of all real polynomials $A(x)$ with leading coefficient 1 such that there exists a real polynomial $B(x)$ that satisfies

$$\frac{1}{A(x)} + \frac{1}{B(x)} + \frac{1}{x+10} = \frac{1}{x}$$

for all real numbers x for which $A(x) \neq 0$, $B(x) \neq 0$, and $x \neq -10, 0$. Compute $\displaystyle\sum_{A \in \mathcal{S}} A(10)$.

7 Individual Round Answers

Answer 1. 63

Answer 2. 60

Answer 3. 2302

Answer 4. $\frac{14}{5}$ (or $2\frac{4}{5}$ or 2.8)

Answer 5. 99956

Answer 6. 8

Answer 7. 508

Answer 8. $5\sqrt{10}$

Answer 9. 49

Answer 10. 46750

8 Individual Round Solutions

Problem 1. Let $N = 888{,}888 \times 9{,}999{,}999$. Compute the sum of the digits of N.

Solution 1. Write N as

$$\begin{aligned} &(10{,}000{,}000 - 1) \cdot 888{,}888 \\ &= 8{,}888{,}880{,}000{,}000 - 888{,}888 \\ &= 8{,}888{,}879{,}111{,}112. \end{aligned}$$

The sum of the digits of N is **63**.

Problem 2. Five equilateral triangles are drawn in the plane so that no two sides of any of the triangles are parallel. Compute the maximum number of points of intersection among all five triangles.

Solution 2. Any two of the triangles intersect in at most six points, because each side of one triangle can intersect the other triangle in at most two points. To count the total number of intersections among the five triangles, note that there are $\binom{5}{2} = 10$ ways to select a pair of triangles, and each pair may result in 6 intersections. Thus $10 \times 6 = 60$ is an upper bound.

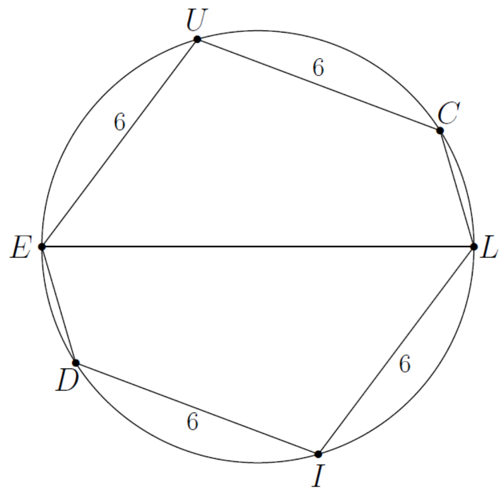
This can be achieved, for example, by taking six equilateral triangles of equal size, centered at a single point, and rotating them different amounts so that no three sides intersect at a single point. Thus the answer is **60**.

Problem 3. Let S be the set of four-digit positive integers for which the sum of the squares of their digits is 17. For example, $2023 \in S$ because $2^2 + 0^2 + 2^2 + 3^2 = 17$. Compute the median of S.

Solution 3. In order for the sums of the squares of four digits to be 17, the digits must be either 0, 2, 2, and 3, or 0, 0, 1, and 4, in some order. If the leading digit is 2, there are $3! = 6$ possible four-digit numbers. If the leading digit is 1, 3, or 4, there are $\frac{3!}{2!} = 3$ possible four-digit numbers. In total, there are $6 + 3 \cdot 3 = 15$ four-digit integers in S, and the median will be the eighth least. The least eight integers in S, from least to greatest, are: 1004, 1040, 1400, 2023, 2032, 2203, 2230, 2302. Thus the median of S is **2302**.

Problem 4. Let $EUCLID$ be a hexagon inscribed in a circle of radius 5. Given that $EU = UC = LI = ID = 6$, and $CL = DE$, compute CL.

Solution 4. Let $CL = x$. Because the quadrilaterals $EUCL$ and $LIDE$ are congruent, $\overline{EL}$ is a diameter of the circle in which the hexagon is inscribed, so $EL = 10$. Furthermore, because $\overline{EL}$ is a diameter of the circle, it follows that the inscribed $\angle EUL$ is a right angle, hence $UL = 8$.



Using Ptolemy's Theorem for cyclic quadrilaterals and the fact that $\triangle ECL$ is also a right triangle,

$$\begin{aligned}
&UC \cdot EL + EU \cdot CL = EC \cdot UL \\
\implies & 6(10+x) = 8\sqrt{100-x^2} \\
\implies & 36(10+x)^2 = 64(10+x)(10-x) \\
\implies & 6\sqrt{10+x} = 8\sqrt{10-x} \\
\implies & 36(10+x) = 64(10-x) \\
\implies & 360 + 36x = 640 - 64x \\
\implies & 100x = 280 \\
\implies & x = \mathbf{\frac{14}{5}}.
\end{aligned}$$

Problem 5. The *ARMLLexicon* consists of 10 letters: $\{A, R, M, L, e, x, i, c, o, n\}$. A *palindrome* is an ordered list of letters that read the same backwards and forwards; for example, *MALAM*, *n*, *oncecno*, and *MoM* are palindromes. Compute the number of 15-letter palindromes that can be spelled using letters in the ARMLLexicon, among which there are four consecutive letters that spell out *ARML*.

Solution 5. Any 15-letter palindrome is determined completely by its first 8 letters, because the last 7 letters must be the first 7 in reverse. Such a palindrome contains the string *ARML* if and only if its first 8 letters contain either *ARML* or *LMRA*. (The string *ARML* cannot cross the middle of the palindrome, because the 7th and 9th letters must be the same.) It therefore suffices to count the number of 8-letter strings consiting of letters in the ARMLLexicon that contain either *ARML* or *LMRA*.

There are 5 possible positions for *ARML*, and likewise with *LMRA*. For each choice of position, there are four remaining letters, which can be any letter in the ARMLLexicon (here, W, X, Y, and Z are used to denote arbitrary letters that need not be distinct). This leads to the following table:

Word	Num. Possibilities
ARMLWXYZ	10^4
WARMLXYZ	10^4
WXARMLYZ	10^4
WXYARMLZ	10^4
WXYZARML	10^4
LMRAWXYZ	10^4
WLMRAXYZ	10^4
WXLMRAYZ	10^4
WXYLMRAZ	10^4
WXYZLMRA	10^4

This gives $10 \cdot 10^4$ possible words, but each word with two of *ARML* or *LMRA* (e.g., *ARMLARML* or *AARMLMRA*) is counted twice. There are four words with two of *ARML* or *LMRA* that use all 8 letters, and four possible types of words that use 7 of the 8 positions and leave one "free space". This leads to the following table:

Word	Num. Possibilities
ARMLARML	1
LMRALMRA	1
ARMLLMRA	1
LMRAARML	1
ARMLMRAW	10
LMRARMLW	10
WARMLMRA	10
WLMRARML	10

Thus the total number of desired words is $10 \cdot 10^4 - 4 \cdot 10 - 4 \cdot 1 = \mathbf{99956}$.

Problem 6. Let 10^y be the product of all real numbers x such that $\log x = \dfrac{3 + \lfloor (\log x)^2 \rfloor}{4}$. Compute y.

Solution 6. First, note that

$$\lfloor (\log x)^2 \rfloor \le (\log x)^2 \implies \frac{3 + \lfloor (\log x)^2 \rfloor}{4} \le \frac{3 + (\log x)^2}{4}.$$

Therefore

$$\log x \le \frac{(\log x)^2 + 3}{4} \implies 0 \le (\log x)^2 - 4\log x + 3 = (\log x - 1)(\log x - 3).$$

This implies either $\log x \le 1$ or $\log x \ge 3$, so $0 \le (\log x)^2 \le 1$ or $(\log x)^2 \ge 9$.

In the first case, $\lfloor (\log x)^2 \rfloor = 0$ or $\lfloor (\log x)^2 \rfloor = 1$, so $\log x = \frac{3}{4}$ or $\log x = 1$, hence $x = 10^{3/4}$ or $x = 10$.

To solve the second case, note that $\lfloor (\log x)^2 \rfloor \ge (\log x)^2 - 1$, so $0 \ge (\log x)^2 - 4\log x + 2$. The solutions to $t^2 - 4t + 2 = 0$ are $t = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$ by the Quadratic Formula, so $2 - \sqrt{2} \le \log x \le 2 + \sqrt{2}$. This implies that $6 - 4\sqrt{2} \le (\log x)^2 \le 6 + 4\sqrt{2}$, so $0 \le \lfloor (\log x)^2 \rfloor \le 11$. However, this case is for $(\log x)^2 \ge 9$, so the only possibilities that need to be considered are $9 \le \lfloor (\log x)^2 \rfloor \le 11$.

- If $\lfloor (\log x)^2 \rfloor = 9$, then $\log x = 3$, so $x = 10^3$.
- If $\lfloor (\log x)^2 \rfloor = 10$, then $\log x = \frac{13}{4}$, so $x = 10^{13/4}$.
- Finally, if $\lfloor (\log x)^2 \rfloor = 11$, then $\log x = \frac{7}{2}$, which yields $(\log x)^2 = \frac{49}{4} > 12$, so there are no solutions.

Thus the product of all possible values of x is $y = 10^{3/4} \cdot 10 \cdot 10^{13/4} \cdot 10^3 = 10^8$, so $y = \mathbf{8}$.

Problem 7. The solutions to the equation $x^2 - 180x + 8 = 0$ are r_1 and r_2. Compute

$$\frac{r_1}{\sqrt[3]{r_2}} + \frac{r_2}{\sqrt[3]{r_1}}.$$

Solution 7. First note that the solutions of the given equation are real because the equation's discriminant is positive. By Vieta's Formulas, $r_1 + r_2 = 180$ $(*)$ and $r_1 r_2 = 8$ $(**)$. The expression to be computed can be written with a common denominator as

$$\frac{\sqrt[3]{r_1^4} + \sqrt[3]{r_2^4}}{\sqrt[3]{r_1 r_2}}.$$

By $(**)$, the denominator is equal to $\sqrt[3]{8} = 2$. To compute the numerator, first let $S_k = \sqrt[3]{r_1^k} + \sqrt[3]{r_2^k}$, so that the numerator is S_4. Then note that

$$\begin{aligned}(S_1)^3 &= r_1 + 3\sqrt[3]{r_1^2 r_2} + 3\sqrt[3]{r_2^2 r_1} + r_2 \\ &= (r_1 + r_2) + 3\sqrt[3]{r_1 r_2}\left(\sqrt[3]{r_1} + \sqrt[3]{r_2}\right) \\ &= 180 + 3 \cdot 2 \cdot S_1, \qquad (\dagger)\end{aligned}$$

where $(*)$ and $(**)$ are used to substitute values into the second equality. Next note that $S_1^3 - 6S_1 - 180$ can be factored as $(S_1 - 6)(S_1^2 + 6S_1 + 30)$. Because the polynomial $t^2 + 6t + 30$ has no real roots, the unique real solution to $(\dagger)$ is $S_1 = 6$, so $\sqrt[3]{r_1} + \sqrt[3]{r_2} = 6$. Square each side of the previous equation to obtain $S_2 + 2\sqrt[3]{r_1 r_2} = 36$, hence $S_2 = 36 - 2 \cdot 2$; that is, $\sqrt[3]{r_1^2} + \sqrt[3]{r_2^2} = 32$. Again, square both sides of this equation to obtain $\sqrt[3]{r_1^4} + 2\sqrt[3]{r_1^2 r_2^2} + \sqrt[3]{r_2^4} = 1024$, so $S_4 + 2\sqrt[3]{r_1^2 r_2^2} = 1024$, from which $S_4 = 1024 - 2 \cdot 4 = 1016$. Thus the desired expression equals $\frac{S_4}{2} = \frac{1016}{2} = \mathbf{508}$.

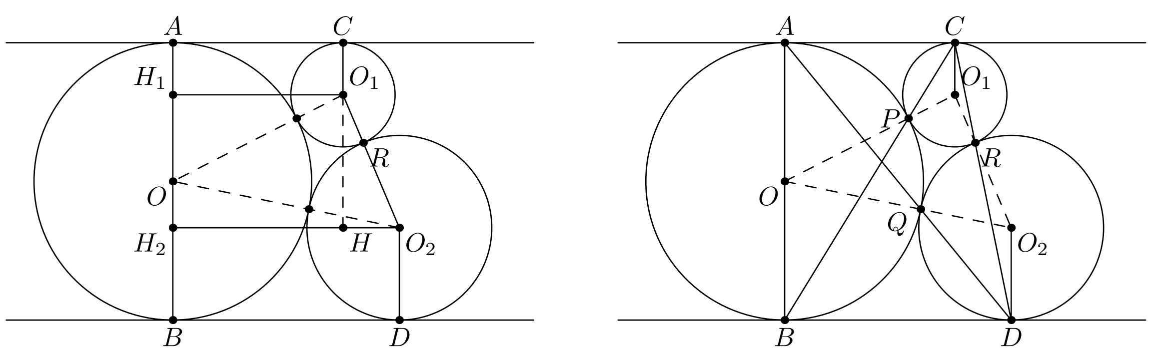
Problem 8. Circle ω is tangent to parallel lines ℓ_1 and ℓ_2 at A and B respectively. Circle ω_1 is tangent to ℓ_1 at C and to ω externally at P. Circle ω_2 is tangent to ℓ_2 at D and to ω externally at Q. Circles ω_1 and ω_2 are also externally tangent to each other. Given that $AQ = 12$ and $DQ = 8$, compute CD.

Solution 8. Let O, O_1 and O_2 be the centers, and let r, r_1 and r_2 be the radii of the circles ω, ω_1, and ω_2, respectively. Let R be the point of tangency between ω_1 and ω_2.

Let H_1 and H_2 be the projections of O_1 and O_2 onto $\overline{AB}$. Also, let H be the projection of O_1 onto $\overline{O_2H_2}$. Note that $OH_1 = r - r_1$, $OH_2 = r - r_2$, $OO_1 = r + r_1$, $OO_2 = r + r_2$, and $O_1O_2 = r_1 + r_2$. From the Pythagorean Theorem, it follows that $O_1H_1 = 2\sqrt{rr_1}$ and $O_2H_2 = 2\sqrt{rr_2}$. Similarly, applying the Pythagorean Theorem to triangle O_1HO_2 yields $(O_1H)^2 + (O_2H)^2 = (O_1O_2)^2$, which is equivalent to

$$\left(2\sqrt{rr_2} - 2\sqrt{rr_1}\right)^2 + (2r - r_1 - r_2)^2 = (r_1 + r_2)^2,$$

which yields $r^2 = 4r_1 r_2$ after simplifying.



Note that $\overline{AO} \parallel \overline{O_2D}$, hence $\angle AOQ \cong \angle DO_2Q$, which implies that isosceles triangles AOQ and DO_2Q are similar. Thus $\angle AQO \cong \angle DQO_2$ and therefore points A, Q, and D are collinear. Analogously, it follows that the points B, P, and C are collinear, as are the points C, R, and D.

In right triangle ABD, $\overline{BQ}$ is the altitude to $\overline{AD}$. By similarity of triangles, it follows that $DQ \cdot DA = BD^2$ and $AQ \cdot AD = AB^2$. Hence $BD = 4\sqrt{10}$, $AB = 4\sqrt{15}$, and $r = 2\sqrt{15}$. Because $\frac{DO_2}{AO} = \frac{DQ}{AQ} = \frac{2}{3}$, it follows that $r_2 = \frac{4}{3}\sqrt{15}$ and $r_1 = \frac{3}{4}\sqrt{15}$.

Note that $AC = 2\sqrt{rr_1} = 3\sqrt{10}$, $BD = 2\sqrt{rr_2} = 4\sqrt{10}$, and

$$CD^2 = AB^2 + (BD - AC)^2 = \left(4\sqrt{15}\right)^2 + \left(4\sqrt{10} - 3\sqrt{10}\right)^2 = 240 + 10 = 250,$$

which implies that $CD = \mathbf{5\sqrt{10}}$.

Alternate Solution: Conclude that $r^2 = 4r_1r_2$, as explained above. Note that $\angle CAQ \cong \angle QDB \cong \angle QRD$, using the fact that the two given lines are parallel and ω_2 is tangent one of them at D. Quadrilateral $CAQR$ is cyclic, so apply Power of a Point to obtain $DQ \cdot DA = DR \cdot DC$. Because $\frac{r_2}{r} = \frac{QD}{QA} = \frac{2}{3}$, conclude that $r_2 = 2x$, $r = 3x$, and hence $r_1 = \frac{9}{8}x$. It follows that $\frac{DR}{CR} = \frac{r_2}{r_1} = \frac{16}{9}$ and $DR = \frac{16}{25} \cdot CD$. Thus

$$DR \cdot DC = \frac{16}{25} \cdot CD^2 = DQ \cdot DA = 8 \cdot 20,$$

hence $CD = 5\sqrt{10}$.

Problem 9. Given quadrilateral $ARML$ with $AR = 20$, $RM = 23$, $ML = 25$, and $AM = 32$, compute the number of different integers that could be the perimeter of $ARML$.

Solution 9. Notice that $\triangle ARM$ is fixed, so the number of integers that could be the perimeter of $ARML$ is the same as the number of integers that could be the length AL in $\triangle ALM$. By the Triangle Inequality, $32 - 25 < AL < 32 + 25$, so AL is at least 8 and no greater than 56. The number of possible integer values for AL is $56 - 8 + 1 = \mathbf{49}$.

Problem 10. Let $\mathcal{S}$ denote the set of all real polynomials $A(x)$ with leading coefficient 1 such that there exists a real polynomial $B(x)$ that satisfies

$$\frac{1}{A(x)} + \frac{1}{B(x)} + \frac{1}{x+10} = \frac{1}{x}$$

for all real numbers x for which $A(x) \neq 0$, $B(x) \neq 0$, and $x \neq -10, 0$. Compute $\sum_{A \in \mathcal{S}} A(10)$.

Solution 10. For brevity, P will be used to represent the polynomial $P(x)$, and let $\deg(P)$ represent the degree of P. Rewrite the given condition as follows:

$$\begin{aligned} \frac{1}{A(x)} + \frac{1}{B(x)} + \frac{1}{x+10} = \frac{1}{x} &\implies \frac{A+B}{AB} = \frac{10}{x(x+10)} \\ &\implies AB - \frac{x(x+10)}{10}A - \frac{x(x+10)}{10}B = 0 \\ &\implies \left(A - \frac{x(x+10)}{10}\right)\left(B - \frac{x(x+10)}{10}\right) = \frac{x^2(x+10)^2}{100}. \end{aligned}$$

Because A and B are both polynomials, $A - \frac{x(x+10)}{10}$ must be some factor F of $\frac{x^2(x+10)^2}{100}$. Furthermore, if $\deg(F) \leq 1$ then A has leading coefficient $\frac{1}{10}$, which is a contradiction. So $\deg(F) \geq 2$. Thus F must be a nonzero constant times one of

$$\{x^2,\ x(x+10),\ (x+10)^2,\ x^2(x+10),\ x(x+10)^2,\ x^2(x+10)^2\}.$$

Because A has leading coefficient 1, it follows that the constant must be 1 if $\deg(F) \geq 3$ and $\frac{9}{10}$ if $\deg(F) = 2$, and thus F is one of

$$\left\{\frac{9}{10}x^2,\ \frac{9}{10}x(x+10),\ \frac{9}{10}(x+10)^2,\ x^2(x+10),\ x(x+10)^2,\ x^2(x+10)^2\right\}.$$

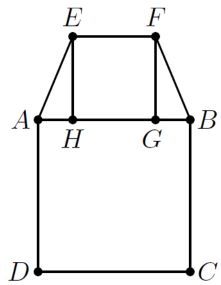
Then

$$\sum A(10) - \frac{10 \cdot 20}{10} = \frac{9}{10} \cdot \left(10^2 + 10 \cdot 20 + 20^2\right) + \left(10^2 \cdot 20 + 10 \cdot 20^2 + 10^2 \cdot 20^2\right) = 46630,$$

so $\sum A(10) = 6 \cdot \frac{10 \cdot 20}{10} + 46630 = \mathbf{46750}$, as desired.

9 Relay Round

Problem 1-1. Square $ABCD$ has side length 22. Points G and H lie on $\overline{AB}$ so that $AH = BG = 5$. Points E and F lie outside square $ABCD$ so that $EFGH$ is a square. Compute the area of hexagon $AEFBCD$.



Problem 1-2. *Let T be the number you will receive.* Let a be the least nonzero digit in T, and let b be the greatest digit in T. In square $NORM$, $NO = b$, and points P_1 and P_2 lie on $\overline{NO}$ and $\overline{OR}$, respectively, so that $OP_1 = OP_2 = a$. A circle centered at O has radius a, and quarter-circular arc $\widehat{P_1P_2}$ is drawn. There is a circle that is tangent to $\widehat{P_1P_2}$ and to sides $\overline{MN}$ and $\overline{MR}$. The radius of this circle can be written in the form $x - y\sqrt{2}$, where x and y are positive integers. Compute $x + y$.

Problem 1-3. *Let T be the number you will receive.* Square $ABCD$ has area T. Points M, N, O, and P lie on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, so that quadrilateral $MNOP$ is a rectangle with $MP = 2$. Compute MN.

Problem 2-1. In a game, a player chooses 2 of the 13 letters from the first half of the alphabet (i.e., A–M) and 2 of the 13 letters from the second half of the alphabet (i.e., N–Z). Aditya plays the game, and then Ayesha plays the game. Compute the probability that Aditya and Ayesha choose the same set of four letters.

Problem 2-2. *Let T be the number you will receive.* Compute the least positive integer n such that when a fair coin is flipped n times, the probability of it landing heads on all n flips is less than T.

Problem 2-3. *Let T be the number you will receive.* Compute the least integer $n > 2023$ such that the equation $x^2 - Tx - n = 0$ has integer solutions.

10 Relay Round Answers

Answer 1-1. 688

Answer 1-2. 36

Answer 1-3. $6\sqrt{2}-2$

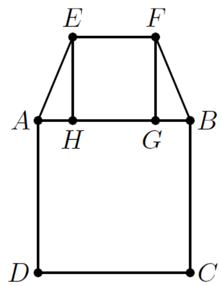
Answer 2-1. $\dfrac{1}{6084}$

Answer 2-2. 13

Answer 2-3. 2028

11 Relay Round Solutions

Problem 1-1. Square $ABCD$ has side length 22. Points G and H lie on $\overline{AB}$ so that $AH = BG = 5$. Points E and F lie outside square $ABCD$ so that $EFGH$ is a square. Compute the area of hexagon $AEFBCD$.

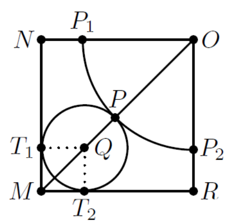


Solution 1-1. Note that $GH = AB - AH - BG = 22 - 5 - 5 = 12$. Thus

$$\begin{aligned}[AEFBCD] &= [ABCD] + [EFGH] + [AEH] + [BFG] \\ &= 22^2 + 12^2 + \tfrac{1}{2} \cdot 5 \cdot 12 + \tfrac{1}{2} \cdot 5 \cdot 12 \\ &= 484 + 144 + 30 + 30 \\ &= \mathbf{688}.\end{aligned}$$

Problem 1-2. *Let T be the number you will receive.* Let a be the least nonzero digit in T, and let b be the greatest digit in T. In square $NORM$, $NO = b$, and points P_1 and P_2 lie on $\overline{NO}$ and $\overline{OR}$, respectively, so that $OP_1 = OP_2 = a$. A circle centered at O has radius a, and quarter-circular arc $\widehat{P_1P_2}$ is drawn. There is a circle that is tangent to $\widehat{P_1P_2}$ and to sides $\overline{MN}$ and $\overline{MR}$. The radius of this circle can be written in the form $x - y\sqrt{2}$, where x and y are positive integers. Compute $x + y$.

Solution 1-2. Let r and Q denote the respective radius and center of the circle whose radius is concerned. Let this circle be tangent to arc $\widehat{P_1P_2}$ at point P, and let it be tangent to sides $\overline{MN}$ and $\overline{MR}$ at points T_1 and T_2, respectively.



Note that Q lies on diagonal $\overline{MO}$ because it is equidistant to $\overline{MN}$ and $\overline{MR}$. Points Q, P, and O must be collinear because the circles centered at Q and O are mutually tangent at point P. It therefore follows that P also lies on diagonal $\overline{MO}$. Because triangles QT_1M and QT_2M are isosceles right triangles, it follows that $MQ = r\sqrt{2}$. Thus

$$b\sqrt{2} = MO = MQ + QP + PO = r\sqrt{2} + r + a.$$

Solving this equation yields $r = a + 2b - (a + b)\sqrt{2}$. With $T = 688$, $a = 6$ and $b = 8$, so $r = 22 - 14\sqrt{2}$, hence $x + y = 22 + 14 = \mathbf{36}$.

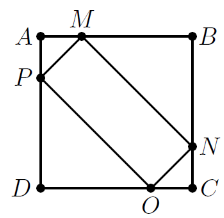
Problem 1-3. *Let T be the number you will receive.* Square $ABCD$ has area T. Points M, N, O, and P lie on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, so that quadrilateral $MNOP$ is a rectangle with $MP = 2$. Compute MN.

Solution 1-3. Let $AM = a$ and $AP = b$, and let $s = \sqrt{T}$ be the side length of square $ABCD$. Then $MB = s - a$ and $DP = s - b$. Using the right angles of $MNOP$ and complementary acute angles in triangles AMP, BNM, CON, and DPO, note that

$$\angle AMP \cong \angle BNM \cong \angle CON \cong DPO.$$

Also note that $\text{m}\angle BMN = 180° - (90° + \text{m}\angle AMP)$, so it also follows that

$$\angle BMN \cong \angle CNO \cong \angle DOP \cong APM.$$



Thus, by side-angle-side congruence, it follows that $\triangle AMP \cong \triangle CON$ and $\triangle BNM \cong \triangle DPO$. Moreover, by side-angle-side similarity, it follows that $\triangle AMP \sim \triangle BNM \sim \triangle CON \sim \triangle DPO$. Thus $BN = s - b$, $NC = b$, $CO = a$, and $OD = s - a$. The similarity relation implies $\frac{AM}{BN} = \frac{AP}{BM}$, so $\frac{a}{s-b} = \frac{b}{s-a}$. Cross-multiplying, rearranging, and simplifying yields $s(a - b) = (a + b)(a - b)$. Thus either $a = b$ or $s = a + b$. In the case where $a = b$, $AM = AP = \frac{2}{\sqrt{2}} = \sqrt{2}$, so $MN = (s - \sqrt{2})\sqrt{2} = s\sqrt{2} - 2$. With $T = 36$, $s = 6$, and the answer is thus $\mathbf{6\sqrt{2} - 2}$. For completeness, it remains to verify that for this particular value of s, the case where $s = a + b$ is impossible. Applying the Pythagorean Theorem in $\triangle MAP$ yields $a^2 + b^2 = 4$. Now if $s = 6 = a + b$, then by squaring, it would follow that $a^2 + b^2 + 2ab = 36 \Longrightarrow 4 + 2ab = 36 \Longrightarrow ab = 16$. But the equation $a + b = a + \frac{16}{a} = 6$ has no real solutions, thus $a + b \neq 6$. (Alternatively, note that by the Arithmetic Mean-Geometric Mean Inequality, $a + \frac{16}{a} \geq 2\sqrt{a \cdot \frac{16}{a}} = 8 > 6$.)

Problem 2-1. In a game, a player chooses 2 of the 13 letters from the first half of the alphabet (i.e., A–M) and 2 of the 13 letters from the second half of the alphabet (i.e., N–Z). Aditya plays the game, and then Ayesha plays the game. Compute the probability that Aditya and Ayesha choose the same set of four letters.

Solution 2-1. The number of ways to choose 2 distinct letters out of 13 is $\frac{13\cdot 12}{2} = 78$. The probability of matching on both halves is therefore $\frac{1}{78^2} = \mathbf{\frac{1}{6084}}$.

Problem 2-2. *Let T be the number you will receive.* Compute the least positive integer n such that when a fair coin is flipped n times, the probability of it landing heads on all n flips is less than T.

Solution 2-2. The problem is equivalent to finding the least integer n such that $\frac{1}{2^n} < T$, or $2^n > \frac{1}{T} = 6084$. Because $2^{12} = 4096$ and $2^{13} = 8192$, the answer is **13**.

Problem 2-3. *Let T be the number you will receive.* Compute the least integer $n > 2023$ such that the equation $x^2 - Tx - n = 0$ has integer solutions.

Solution 2-3. The discriminant of the quadratic, $T^2 + 4n$, must be a perfect square. Because T and the discriminant have the same parity, and the leading coefficient of the quadratic is 1, by the quadratic formula, the discriminant being a perfect square is sufficient to guarantee integer solutions. Before knowing T, note that $\sqrt{4 \cdot 2024} = \sqrt{8096}$ is slightly less than 90 because $90^2 = 8100$, and the square root must have the same parity as T. Because

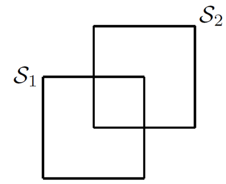
$T = 13$, the square root must be greater than $\sqrt{13^2 + 4 \cdot 2023} = \sqrt{8261}$, which is between 90 and 91, so the desired square root is 91. Hence $13^2 + 4n = 91^2$, so $n = \mathbf{2028}$.

12 Super Relay

1. Compute the least integer greater than 2023, the sum of whose digits is 17.

2. Let T = TNYWR, and let K be the sum of the digits of T. Let r and s be the two roots of the polynomial $x^2 - 18x + K$. Compute $|r - s|$.

3. Let T = TNYWR. Two coplanar squares $\mathcal{S}_1$ and $\mathcal{S}_2$ each have area T and are arranged as shown to form a nonconvex octagon. The center of $\mathcal{S}_1$ is a vertex of $\mathcal{S}_2$, and the center of $\mathcal{S}_2$ is a vertex of $\mathcal{S}_1$. Compute $\dfrac{\text{area of the union of } \mathcal{S}_1 \text{ and } \mathcal{S}_2}{\text{area of the intersection of } \mathcal{S}_1 \text{ and } \mathcal{S}_2}$.



4. Let T = TNYWR, and let $K = 9T$. Let $A_1 = 2$, and for $n \geq 2$, let

$$A_n = \begin{cases} A_{n-1} + 1 & \text{if } n \text{ is not a perfect square,} \\ \sqrt{n} & \text{if } n \text{ is a perfect square.} \end{cases}$$

Compute A_K.

5. Let T = TNYWR. The number $20^T \cdot 23^T$ has K positive divisors. Compute the greatest prime factor of K.

6. Let T = TNYWR. Compute the positive integer $n \neq 17$ for which $\dbinom{T-3}{17} = \dbinom{T-3}{n}$.

7. Let T = TNYWR. Compute the units digit of $T^{2023} + T^{20} - T^{23}$.

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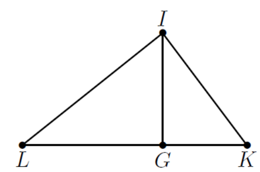
15. In acute triangle ILK, shown in the figure, point G lies on $\overline{LK}$ so that $\overline{IG} \perp \overline{LK}$. Given that $IL = \sqrt{41}$ and $LG = IK = 5$, compute GK.



14. Let T = TNYWR. Suppose that T fair coins are flipped. Compute the probability that at least one tails is flipped.

13. Let T = TNYWR. The number T can be expressed as a reduced fraction $\frac{m}{n}$, where m and n are positive integers whose greatest common divisor is 1. The equation $x^2 + (m+n)x + mn = 0$ has two distinct real solutions. Compute the lesser of these two solutions.

12. Let T = TNYWR, and let $i = \sqrt{-1}$. Compute the positive integer k for which $(-1+i)^k = \dfrac{1}{2^T}$.

11. Let T = TNYWR. Compute the value of x that satisfies $\log_4 T = \log_2 x$.

10. Let T = TNYWR. Pyramid $LEOJS$ is a right square pyramid with base $EOJS$, whose area is T. Given that $LE = 5\sqrt{2}$, compute $[LEO]$.

9. Let T = TNYWR. Compute the units digit of $T^{2023} + (T-2)^{20} - (T+10)^{23}$.

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8. Let r and R be the lesser and greater numbers, respectively, of the two numbers you will receive. A circle with radius r is centered at A, and a circle with radius R is centered at B. The two circles are internally tangent. Point P lies on the smaller circle so that $\overline{BP}$ is tangent to the smaller circle. Compute BP.

13 Super Relay Answers

1. 2069

2. 16

3. 7

4. 21

5. 43

6. 23

7. 1

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15. 3

14. $\frac{7}{8}$

13. -8

12. 16

11. 4

10. 7

9. 5

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8. $\sqrt{15}$

14 Super Relay Solutions

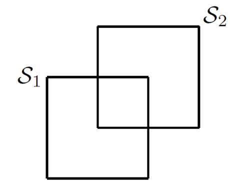
Problem 1. Compute the least integer greater than 2023, the sum of whose digits is 17.

Solution 1. A candidate for desired number is $\underline{2}\,\underline{0}\,\underline{X}\,\underline{Y}$, where X and Y are digits and $X+Y=15$. To minimize this number, take $Y=9$. Then $X=6$, and the desired number is **2069**.

Problem 2. Let $T=$ TNYWR, and let K be the sum of the digits of T. Let r and s be the two roots of the polynomial $x^2-18x+K$. Compute $|r-s|$.

Solution 2. Note that $|r-s|=\sqrt{r^2-2rs+s^2}=\sqrt{(r+s)^2-4rs}$. By Vieta's Formulas, $r+s=-(-18)$ and $rs=K$, so $|r-s|=\sqrt{18^2-4K}$. With $T=2069$, $K=17$, and the answer is $\sqrt{324-68}=\sqrt{256}=\mathbf{16}$.

Problem 3. Let $T=$ TNYWR. Two coplanar squares $\mathcal{S}_1$ and $\mathcal{S}_2$ each have area T and are arranged as shown to form a nonconvex octagon. The center of $\mathcal{S}_1$ is a vertex of $\mathcal{S}_2$, and the center of $\mathcal{S}_2$ is a vertex of $\mathcal{S}_1$. Compute $\dfrac{\text{area of the union of } \mathcal{S}_1 \text{ and } \mathcal{S}_2}{\text{area of the intersection of } \mathcal{S}_1 \text{ and } \mathcal{S}_2}$.



Solution 3. Let $2x$ be the side length of the squares. Then the intersection of $\mathcal{S}_1$ and $\mathcal{S}_2$ is a square of side length x, so its area is x^2. The area of the union of $\mathcal{S}_1$ and $\mathcal{S}_2$ is $(2x)^2+(2x)^2-x^2=7x^2$. Thus the desired ratio of areas is $\frac{7x^2}{x^2}=\mathbf{7}$ (independent of T).

Problem 4. Let $T=$ TNYWR, and let $K=9T$. Let $A_1=2$, and for $n\geq 2$, let

$$A_n=\begin{cases} A_{n-1}+1 & \text{if } n \text{ is not a perfect square,}\\ \sqrt{n} & \text{if } n \text{ is a perfect square.}\end{cases}$$

Compute A_K.

Solution 4. Let $\lfloor\sqrt{n}\rfloor=x$. Then n can be written as x^2+y, where y is an integer such that $0\leq y<2x+1$. Let m be the greatest perfect square less than or equal to $9T$. Then the definition of the sequence and the previous observation imply that $A_K=A_{9T}=\sqrt{m}+(9T-m)=\lfloor\sqrt{9T}\rfloor+\left(9T-\lfloor\sqrt{9T}\rfloor^2\right)$. With $T=7$, $K=9T=63$, $\lfloor\sqrt{9T}\rfloor=7$, and the answer is therefore $7+(63-7^2)=\mathbf{21}$.

Problem 5. Let $T=$ TNYWR. The number $20^T\cdot 23^T$ has K positive divisors. Compute the greatest prime factor of K.

Solution 5. Write $20^T\cdot 23^T$ as $2^{2T}\cdot 5^T\cdot 23^T$. This number has $K=(2T+1)(T+1)^2$ positive divisors. With $T=21$, $K=43\cdot 22^2$. The greatest prime factor of K is **43**.

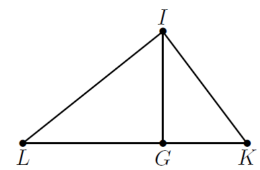
Problem 6. Let $T=$ TNYWR. Compute the positive integer $n\neq 17$ for which $\binom{T-3}{17}=\binom{T-3}{n}$.

Solution 6. Using the symmetry property of binomial coefficients, the desired value of n is $T-3-17=T-20$. With $T=43$, the answer is **23**.

Problem 7. Let $T = \text{TNYWR}$. Compute the units digit of $T^{2023} + T^{20} - T^{23}$.

Solution 7. Assuming that T is a positive integer, because units digits of powers of T cycle in groups of at most 4, the numbers T^{2023} and T^{23} have the same units digit, hence the number $T^{2023} - T^{23}$ has a units digit of 0, and the answer is thus the units digit of T^{20}. With $T = 23$, the units digit of 23^{20} is the same as the units digit of 3^{20}, which is the same as the units digit of $3^4 = 81$, so the answer is **1**.

Problem 15. In acute triangle ILK, shown in the figure, point G lies on $\overline{LK}$ so that $\overline{IG} \perp \overline{LK}$. Given that $IL = \sqrt{41}$ and $LG = IK = 5$, compute GK.



Solution 15. Using the Pythagorean Theorem, $IG = \sqrt{(IL)^2 - (LG)^2} = \sqrt{41 - 25} = 4$, and $GK = \sqrt{(IK)^2 - (IG)^2} = \sqrt{25 - 16} = \mathbf{3}$.

Problem 14. Let $T = \text{TNYWR}$. Suppose that T fair coins are flipped. Compute the probability that at least one tails is flipped.

Solution 14. The probability of flipping all heads is $(\frac{1}{2})^T$, so the probability of flipping at least one tails is $1 - \frac{1}{2^T}$. With $T = 3$, the desired probability is $1 - \frac{1}{8} = \mathbf{\frac{7}{8}}$.

Problem 13. Let $T = \text{TNYWR}$. The number T can be expressed as a reduced fraction $\frac{m}{n}$, where m and n are positive integers whose greatest common divisor is 1. The equation $x^2 + (m + n)x + mn = 0$ has two distinct real solutions. Compute the <u>lesser</u> of these two solutions.

Solution 13. The left-hand side of the given equation can be factored as $(x + m)(x + n)$. The two solutions are therefore $-m$ and $-n$, so the answer is $\min\{-m, -n\}$. With $T = \frac{7}{8}$, $m = 7$, $n = 8$, and $\min\{-7, -8\}$ is $\mathbf{-8}$.

Problem 12. Let $T = \text{TNYWR}$, and let $i = \sqrt{-1}$. Compute the positive integer k for which $(-1 + i)^k = \dfrac{1}{2^T}$.

Solution 12. Note that $(-1+i)^2 = 1+2i-1 = 2i$. Thus $(-1+i)^4 = (2i)^2 = -4$, and $(-1+i)^8 = (-4)^2 = 16$. The expression $\frac{1}{2^T}$ is a power of 16 if T is a negative multiple of 4. With $T = -8$, $\frac{1}{2^{-8}} = 2^8 = 16^2 = \left((-1+i)^8\right)^2 = (-1+i)^{16}$, so the desired value of k is **16**.

Problem 11. Let $T = \text{TNYWR}$. Compute the value of x that satisfies $\log_4 T = \log_2 x$.

Solution 11. By the change of base rule and a property of logs, $\log_4 T = \frac{\log_2 T}{\log_2 4} = \frac{\log_2 T}{2} = \log_2 \sqrt{T}$. Thus $x = \sqrt{T}$, and with $T = 16$, $x = \mathbf{4}$.

Problem 10. Let $T = \text{TNYWR}$. Pyramid $LEOJS$ is a right square pyramid with base $EOJS$, whose area is T. Given that $LE = 5\sqrt{2}$, compute $[LEO]$.

Solution 10. Let the side length of square base $EOJS$ be $2x$, and let M be the midpoint of $\overline{EO}$. Then $\overline{LM} \perp \overline{EO}$, and $LM = \sqrt{\left(5\sqrt{2}\right)^2 - x^2}$ by the Pythagorean Theorem. Thus $[LEO] = \frac{1}{2} \cdot 2x\sqrt{\left(5\sqrt{2}\right)^2 - x^2} =$

$x\sqrt{\left(5\sqrt{2}\right)^2 - x^2}$. With $T = 4$, $x = 1$, and the answer is $1 \cdot \sqrt{50-1} = \mathbf{7}$.

Problem 9. Let $T = \text{TNYWR}$. Compute the units digit of $T^{2023} + (T-2)^{20} - (T+10)^{23}$.

Solution 9. Note that T and $T+10$ have the same units digit. Because units digits of powers of T cycle in groups of at most 4, the numbers T^{2023} and $(T+10)^{23}$ have the same units digit, hence the number $T^{2023} - (T+10)^{23}$ has a units digit of 0, and the answer is thus the units digit of $(T-2)^{20}$. With $T = 7$, the units digit of 5^{20} is **5**.

Problem 8. Let r and R be the lesser and greater numbers, respectively, of the two numbers you will receive. A circle with radius r is centered at A, and a circle with radius R is centered at B. The two circles are internally tangent. Point P lies on the smaller circle so that $\overline{BP}$ is tangent to the smaller circle. Compute BP.

Solution 8. Draw radius AP and note that APB is a right triangle with $\mathrm{m}\angle APB = 90^\circ$. Note that $AB = R - r$ and $AP = r$, so by the Pythagorean Theorem, $BP = \sqrt{(R-r)^2 - r^2} = \sqrt{R^2 - 2Rr}$. With $r = 1$ and $R = 5$, it follows that $BP = \boldsymbol{\sqrt{15}}$.

15 Tiebreaker Round

Problem 1. For an integer $n \geq 4$, define a_n to be the product of all real numbers that are roots to at least one quadratic polynomial whose coefficients are positive integers that sum to n. Compute

$$\frac{a_4}{a_5} + \frac{a_5}{a_6} + \frac{a_6}{a_7} + \cdots + \frac{a_{2022}}{a_{2023}}.$$

Problem 2. Suppose that u and v are distinct numbers chosen at random from the set $\{1, 2, 3, \ldots, 30\}$. Compute the probability that the roots of the polynomial $(x + u)(x + v) + 4$ are integers.

Problem 3. The degree-measures of the interior angles of convex hexagon $TIEBRK$ are all integers in arithmetic progression. Compute the least possible degree-measure of the smallest interior angle in hexagon $TIEBRK$.

16 Tiebreaker Round Answers

Answer 1. -2019

Answer 2. $\dfrac{17}{145}$

Answer 3. 65 (or 65°)

17 Tiebreaker Round Solutions

Problem 1. For an integer $n \geq 4$, define a_n to be the product of all real numbers that are roots to at least one quadratic polynomial whose coefficients are positive integers that sum to n. Compute

$$\frac{a_4}{a_5} + \frac{a_5}{a_6} + \frac{a_6}{a_7} + \cdots + \frac{a_{2022}}{a_{2023}}.$$

Solution 1. For an integer $n \geq 4$, let S_n denote the set of real numbers x that are roots to at least one quadratic polynomial whose coefficients are positive integers that sum to n. (Note that S_n is nonempty, as the polynomial $x^2 + (n-2)x + 1$ has a discriminant of $(n-2)^2 - 4$, which is nonnegative for $n \geq 4$.) Then $a_n = \prod_{x \in S_n} x$.

Suppose that a, b, and c are positive integers and x is a real solution to $ax^2 + bx + c = 0$. Then x must be nonzero. (In fact, x must be negative.) Dividing the above equation by x^2 yields $a + \frac{b}{x} + \frac{c}{x^2} = 0$, thus $r = \frac{1}{x}$ is a solution to the quadratic equation $cr^2 + br + a = 0$. This shows that $x \in S_n$ if and only if $\frac{1}{x} \in S_n$.

One might then think that a_n must equal 1, because one can presumably pair up all elements in a given S_n into $\left\{x, \frac{1}{x}\right\}$ pairs. But there is a (negative) value of x for which $x = \frac{1}{x}$, namely $x = -1$. Therefore the value of a_n depends only on whether $-1 \in S_n$. It is readily seen via a parity argument that $-1 \in S_n$ if and only if n is even. If $n = 2k$, then the polynomial $x^2 + kx + (k-1)$ has -1 as a root. (In fact, any quadratic polynomial whose middle coefficient is k and whose coefficients sum to $2k$ will work.) But if $n = 2k+1$, then $a(-1)^2 + b(-1) + c = a - b + c = (a+b+c) - 2b = (2k+1) - 2b$ will be odd, and so $-1 \notin S_n$.

Thus $a_n = -1$ when n is even, $a_n = 1$ when n is odd, and finally,

$$\frac{a_4}{a_5} + \frac{a_5}{a_6} + \frac{a_6}{a_7} + \cdots + \frac{a_{2022}}{a_{2023}} = \underbrace{(-1) + (-1) + (-1) + \cdots + (-1)}_{2019\ (-1)\text{s}} = \mathbf{-2019}.$$

Problem 2. Suppose that u and v are distinct numbers chosen at random from the set $\{1, 2, 3, \ldots, 30\}$. Compute the probability that the roots of the polynomial $(x+u)(x+v)+4$ are integers.

Solution 2. Assume without loss of generality that $u > v$. The condition that $(x+u)(x+v)+4$ has integer roots is equivalent to the discriminant $(u+v)^2 - 4(uv+4) = (u-v)^2 - 16$ being a perfect square. This is possible if and only if $u - v = 4$ or $u - v = 5$. There are $(30-4) + (30-5) = 26 + 25 = 51$ such ordered pairs (u, v), so the answer is

$$\frac{51}{\binom{30}{2}} = \mathbf{\frac{17}{145}}.$$

Problem 3. The degree-measures of the interior angles of convex hexagon $TIEBRK$ are all integers in arithmetic progression. Compute the least possible degree-measure of the smallest interior angle in hexagon $TIEBRK$.

Solution 3. The sum of the measures of the interior angles of a convex hexagon is $(6-2)(180^\circ) = 720^\circ$. Let the measures of the angles be a, $a+d$, $\ldots$, $a+5d$. This implies that $6a+15d = 720 \to 2a+5d = 240 \to 5d = 240-2a$. Note that $a + 5d < 180 \to 240 - a < 180 \to a > 60$. By inspection, note that the least a greater than 60 that produces an integer d is $a = 65 \to d = 22$. Thus the least possible degree-measure of the smallest angle is $\mathbf{65^\circ}$, and the hexagon has angles with degree-measures 65°, 87°, 109°, 131°, 153°, and 175°.